

# Distributional Itô's Formula and Regularization of Generalized Wiener Functionals

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**Abstract.** We investigate Bochner integrabilities of generalized Wiener functionals. We further formulate an Itô formula for a diffusion in a distributional setting, and apply to investigate differentiability-index  $s$  and integrability-index  $p \geq 2$  for which the Bochner integral belongs to  $\mathbb{D}_p^s$ .

## 1. INTRODUCTION

In this paper, we justify the symbol “ $\int_0^T \delta_y(X_t)dt$ ” denoting a quantity relating to the local time of a  $d$ -dimensional diffusion process  $X = (X_t)_{t \geq 0}$  with  $X_0$  being deterministic (in multi-dimensional case, we assume  $X_0 \neq y$ ), or more generally, the object “ $\int_0^T \Lambda(X_t)dt$ ” where  $\Lambda$  is a distribution. Our diffusion process  $X = (X_t)_{t \geq 0}$  is assumed to satisfy a  $d$ -dimensional stochastic differential equation

$$dX_t = \sigma(X_t)dw(t) + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d,$$

where  $w = (w^1(t), \dots, w^d(t))_{t \geq 0}$  is a  $d$ -dimensional Wiener process with  $w(0) = 0$ . The main conditions on  $\sigma = (\sigma_j^i)_{1 \leq i, j \leq d}$  and  $b = (b^i)_{1 \leq i \leq d}$  under which we will work are combinations from the following.

**Hypothesis 1.1.** (H1) the coefficients  $\sigma$  and  $b$  are  $C^\infty$ , and have bounded derivatives in all order  $\geq 1$ .

(H2)  $(\sigma^* \sigma)(x)$  is strictly positive, where  $x = X_0$  and  $\sigma^*$  is the transposed matrix of  $\sigma$ .

(H3) There exists  $\lambda > 0$  such that

$$\lambda |\xi|_{\mathbb{R}^d}^2 \leq \langle \xi, (\sigma^* \sigma)(x) \xi \rangle_{\mathbb{R}^d} \quad \text{for all } \xi \in \mathbb{R}^d,$$

where  $\langle \bullet, \bullet \rangle_{\mathbb{R}^d}$  is the standard inner product on  $\mathbb{R}^d$ , and  $|\bullet|_{\mathbb{R}^d} = |\bullet|$  is the corresponding norm.

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(H4) There exists  $\kappa > 0$  such that

$$\langle \xi, (\sigma^* \sigma)(x) \xi \rangle_{\mathbb{R}^d} \leq \kappa |\xi|_{\mathbb{R}^d}^2 \quad \text{for all } \xi \in \mathbb{R}^d.$$

We further formulate stochastic integrals and an Itô formula in this distributional setting and investigate the differentiability-index  $s$  and the integrability-index  $p \geq 2$  for which the local time  $\int_0^T \delta_y(X_t) dt$  belongs to  $\mathbb{D}_2^s$ .

In fact, we will formulate  $\int_0^T \delta_y(X_t) dt$  as a Bochner integral in the space of generalized Wiener functional. We remark here the Bochner integrability seems to be not so trivial when  $y = X_0$  since  $\delta_y(X_t)$  no longer makes sense at  $t = 0$ . On the other hand, the local time is usually formulated as a classical Wiener functional, and hence once the Bochner integrability is proved, a “smoothing effect” should occur in the Bochner integral  $\int_0^T \delta_y(X_t) dt$ , i.e., the differentiability-index for  $\int_0^T \delta_y(X_t) dt$ , should be greater than that of  $\delta_y(X_t)$ .

In the Brownian case:  $X_t = w(t)$ , everything can be explicitly computed and we can exhibit this phenomenon. Namely, the following is the prototype of this study.

Let  $\mathcal{S}'(\mathbb{R}^d)$  denote the space of all Schwartz distributions on  $\mathbb{R}^d$ .

**Theorem 1.1.** *Assume  $d = 1$ . Let  $\Lambda \in \mathcal{S}'(\mathbb{R})$  and  $s \in \mathbb{R}$ . If  $\Lambda(w(T)) \in \mathbb{D}_2^s$  then the mapping*

$$(0, T] \ni t \mapsto \sqrt{\frac{T}{t}} \Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) \in \mathbb{D}_2^s$$

*is Bochner integrable in  $\mathbb{D}_2^s$  and we have*

$$\int_0^T \sqrt{\frac{T}{t}} \Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) dt \in \mathbb{D}_2^{s+1}.$$

Since it is known that  $\delta_0(w(t)) \in \mathbb{D}_2^{(-1/2)-}$  (see Watanabe [27]), we obtain  $\int_0^T \delta_0(w(t)) dt \in \mathbb{D}_2^{(1/2)-}$ , which agrees with the result by Nualart-Vives [16] and Watanabe [28]. The proof of this theorem is due to the chaos computations (which is essentially the same as [16] but with *no* approximations of the integrand). This computation brings as by-product a Hölder continuity result of the local time with respect to space variable, associated to the Brownian motion on a Riemannian manifold  $\mathbb{R}$ , i.e., the case of  $b = \sigma \sigma' / 2$ :

**Theorem 1.2.** *Let  $d = 1$ . Assume (H1), (H3) and that the drift-coefficient is given by  $b = \sigma \sigma' / 2$ . Then for each  $s < \frac{1}{2}$  and  $\beta \in (0, \min\{\frac{1}{2} - s, 1\})$ , there exists a constant  $c = c(s, \beta) > 0$  such that*

$$\left\| \sigma(y) \int_0^1 \delta_y(X_t) dt - \sigma(z) \int_0^1 \delta_z(X_t) dt \right\|_{2,s} \leq c |y - z|^\beta$$

*for every  $y, z \in \mathbb{R}$ .*

The proof of this theorem is seemingly interesting in its own right. The study of Hölder continuity of local times had been initiated by Trotter [22, inequalities (2.1) and (2.3)], in which the almost-sure Hölder-continuity of the Brownian local time  $\{l(t, x) : t \geq 0, x \in \mathbb{R}\}$  in time-space variable  $(t, x)$  was proved (see also Boufoussi-Roynette [6]). There are a lot of such studies (see, e.g., Liang [15], Ait Ouahra-Kissami-Ouahhabi [3], Shuwen-Cheng [19] and references therein), but we do not enter anymore.

Theorem 1.2 implies immediately

**Corollary 1.3.** *Under the conditions in Theorem 1.2, let  $p_t(x, y)$  be the transition density of  $X_t$ . Then the mapping*

$$\mathbb{R} \ni y \mapsto \sigma(y) \int_0^1 p_t(x, y) dt \in \mathbb{R}$$

*is (globally)  $\beta$ -Hölder continuous for every  $\beta < 1$ .*

The latter half of this paper concerns with an Itô formula in a distributional setting. The classical Itô-Tanaka formula had been extended with several formulations (see Föllmer-Protter-Shiryayev [8], Bouleau-Yor [7], Wang [25], Kubo [13] and so on). In particular, according to results in [25] and [7], the Itô-Tanaka formula for  $f(X_t)$  is valid in the case where  $f$  is just a convex function. In our case, we obtained

**Theorem 1.4.** *Assume (H1) and (H2). Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a measurable function such that*

- (i)  *$f$  is continuous at  $x$ ,*
- (ii)  *$f$  is at most exponential growth,*
- (iii)  *$\int_0^T \|(A_i f)(X_t)\|_{2, -k}^2 dt < +\infty$  for  $i = 1, \dots, d$ ,*
- (iv)  *$\int_0^T \|(L f)(X_t)\|_{2, -k} dt < +\infty$*

*for some  $k \in \mathbb{N}$ . Then we have*

$$f(X_T) - f(x) = \sum_{i=1}^d \int_0^T (A_i f)(X_t) dw^i(t) + \int_0^T (L f)(X_t) dt \quad \text{in } \mathbb{D}^{-\infty}.$$

Here,  $A_i = \sum_{j=1}^d \sigma_i^j \partial / (\partial x_j)$  and  $L$  is the generator of the diffusion process  $X$ . The definition of stochastic integral will be given in Subsection 4.1 and the time-integral  $\int_0^T (L f)(X_t) dt$  is understood in the sense of Bochner integral in  $\mathbb{D}_2^{-k}$ . Kubo [13] also obtained an Itô formula for Brownian motion in a distributional setting, though his formula does not need to consider the Bochner integrability because the time-interval of integration is a closed interval excluding zero. A generalization to one-dimensional fractional Brownian motion case was done by Bender [4, and see references therein], in which, even the case where the time-interval of integration is such as  $(0, T]$  is considered ([4, Theorem 4.4]), though the first distributional derivative of  $f$  is assumed to be a regular distribution. But he did not give a systematic treatment about

Bochner integrability. Theorem 1.4 will be proved in Section 4 and it will be established in Subsection 4.2 even the case where  $f$  itself is a distribution of exponential-type and furthermore the time-interval of integration is  $(0, T]$ .

A distribution  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$  is said to be *positive* if  $\langle \Lambda, f \rangle \geq 0$  for every nonnegative test function  $f \in \mathcal{S}(\mathbb{R}^d)$ . To include local times for diffusions in our scope, we prepare the following

**Theorem 1.5.** *Assume  $d = 1$ , (H1) and (H2). Let  $\Lambda \in \mathcal{S}'(\mathbb{R})$  be positive. Then for every  $p \in (1, \infty)$ , we have  $\int_0^T \|\Lambda(X_t)\|_{p, -2} dt < +\infty$ .*

Hence the mapping  $(0, T] \ni t \mapsto \delta_y(X_t) \in \mathbb{D}_p^{-2}$  is Bochner integrable in the case of  $d = 1$ . For multi-dimensional cases, it is sufficient to assume  $x \neq y$  in order to guarantee the Bochner integrability (Proposition 3.10).

Finally, let  $H_p^s(\mathbb{R}^d) := (1 - \Delta)^{-s/2} L_p(\mathbb{R}^d, dz)$ ,  $p \in (1, \infty)$ ,  $s \in \mathbb{R}$  be the Bessel potential spaces (see [1] or [12] for details). We will then apply an Itô formula (Theorem 4.7) to derive the following.

**Corollary 1.6.** *Assume (H1), (H3) and (H4). Let  $p \in (1, \infty)$ ,  $s \in \mathbb{R}$ . Then for each  $\Lambda \in H_p^s(\mathbb{R}^d)$ , we have*

- (i)  $\Lambda(X_t) \in \mathbb{D}_{p'}^s$  for  $t > 0$  and  $p' \in (1, p)$ ;
- (ii) if  $p > 2$ , we further have  $\int_{t_0}^T \Lambda(X_t) dt \in \mathbb{D}_{p'}^{s+1}$  for  $t_0 \in (0, T]$  and  $p' \in [2, p)$ .

It might be natural to ask about the class to which  $\int_{t_0}^T \Lambda(X_t) dt$  belongs when  $t_0 = 0$ . Some examples are included in Subsection 4.3.

The organization of the current paper is as follows: We first review the classical Malliavin calculus in Section 2 to introduce several notations. In particular, the mapping of Watanabe's pull-back will be extended to the space of distributions of exponential-type. Section 3 is devoted to investigate Bochner integrability of the mapping  $(0, T] \ni t \mapsto \Lambda(X_t)$  where  $\Lambda$  is a distribution. We will illustrate the Brownian case with detailed computations. The know-how there bring by-product a Hölder continuity in the space variable of the local time in the case where the stochastic differential equation is written in a Fisk-Stratonovich symmetric form. In Section 4, we give a definition of stochastic integrals and formulate an Itô formula in this distributional setting. Corollary 1.6 and some examples will be presented in Subsection 4.3. Several estimates necessary for these examples are wrapped up in Appendix A.

## 2. REVIEW OF MALLIAVIN CALCULUS

First, we make a brief review of the classical Malliavin calculus over the  $d$ -dimensional classical Wiener space to introduce notations.

Let  $(W, \mathcal{F}, \mathbf{P})$  be the  $d$ -dimensional Wiener space on  $[0, T]$ , that is,  $W$  is the space of all continuous functions  $[0, T] \rightarrow \mathbb{R}^d$ ,  $\mathcal{F}$  is the  $\sigma$ -field generated by the canonical process  $W \ni w \mapsto w(t) \in \mathbb{R}^d$ ,  $0 \leq t \leq T$ , and  $\mathbf{P}$  is the Wiener measure starting from zero. The expectation under  $\mathbf{P}$  will be denoted by  $\mathbf{E}$ . The space  $W$  contains the subspace  $H$ , consisting of all absolutely continuous  $h \in W$  with  $h(0) = 0$  and the square-integrable derivative. The subspace  $H$  is called the *Cameron-Martin subspace* and forms a real Hilbert space under the inner product

$$\langle h_1, h_2 \rangle_H := \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle_{\mathbb{R}^d} dt, \quad h_1, h_2 \in H.$$

It is known that  $L_2 := L_2(W, \mathcal{F}, \mathbf{P})$  has the following orthogonal decomposition, called the *Wiener-Itô chaos expansion*:

$$L_2 = \mathbb{R} \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots,$$

where each  $\mathcal{C}_k$  is a closed linear subspace of  $L_2$  spanned by multiple stochastic integrals

$$\int_0^T \cdots \int_0^T \langle \dot{h}_1(t_1) \otimes \cdots \otimes \dot{h}_k(t_k), dw(t_1) \otimes \cdots \otimes dw(t_k) \rangle_{(\mathbb{R}^d)^{\otimes k}},$$

for  $h_1, \dots, h_k \in H$ , of  $k$ -th degree. Each  $\mathcal{C}_k$  is called the subspace of *Wiener's homogeneous chaos of  $k$ -th order*. We denote by  $J_k$  the orthogonal projection onto  $\mathcal{C}_k$ . For each separable Hilbert space  $(E, \langle \bullet, \bullet \rangle_E)$ ,  $L_p(E)$  denotes the space of  $E$ -valued  $p$ -th integrable random variables  $F$ , with norm  $\|F\|_p = \mathbf{E}[|F|_E^p]^{1/p}$ . Each projection  $J_n$  extends to  $L_2(E) \cong L_2 \otimes E \rightarrow \mathcal{C}_n \otimes E$ , which is still denoted by the same symbol.

For each  $s \in \mathbb{R}$  and  $p \in (1, \infty)$ , a Sobolev-type space  $\mathbb{D}_p^s(E)$  (we write this  $\mathbb{D}_p^s$  when  $E = \mathbb{R}$ ) is defined as the completion of  $\mathcal{P} := \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} \{F \in L_2(E) : J_m F = 0\}$  under the norm  $\|\cdot\|_{p,s}$  defined by  $\|F\|_{p,s} = \|(I - \mathcal{L})^{s/2} F\|_p$  for  $F \in \mathcal{P}$ , where

$$(2.1) \quad (I - \mathcal{L})^{s/2} F = \sum_{k=0}^{\infty} (1+k)^{s/2} J_k F, \quad F \in \mathcal{P}.$$

Note that  $\mathbb{D}_p^0 = L_p$  for  $p \in (1, \infty)$ , and

$$(2.2) \quad \|F\|_{2,s}^2 = \sum_{k=0}^{\infty} (1+k)^s \|J_k F\|_2^2, \quad F \in \mathbb{D}_2^s.$$

We further define

$$\mathbb{D}^{\infty}(E) := \bigcap_{s>0} \bigcap_{1<p<\infty} \mathbb{D}_p^s(E) \quad \text{and} \quad \mathbb{D}^{-\infty}(E) := \bigcup_{s<0} \bigcup_{1<p<\infty} \mathbb{D}_p^s(E).$$

It is known that  $(\mathbb{D}_p^s(E))' = \mathbb{D}_q^{-s}(E)$  iff  $1/p + 1/q = 1$  (where “ $\prime$ ” stands for the “continuous dual”) for each  $s \in \mathbb{R}$ , the space  $\mathbb{D}^{\infty}(E)$  is a complete countably normed space and  $\mathbb{D}^{-\infty}(E)$  is its dual which is called the space of *generalized Wiener functionals*. The pairing of  $\Phi \in$

$\mathbb{D}^{-\infty}(E)$  and  $F \in \mathbb{D}^\infty(E)$  is written as  $\mathbf{E}[\Phi F] := \mathbb{D}^{-\infty}(E) \langle \Phi, F \rangle_{\mathbb{D}^\infty(E)}$ , and then  $\mathbf{E}[\Psi] = \mathbb{D}^{-\infty} \langle \Psi, 1 \rangle_{\mathbb{D}^\infty}$  is called the *generalized expectation* of  $\Psi \in \mathbb{D}^{-\infty}$ .

One can define a (continuous) linear operator  $D : \mathbb{D}^{-\infty}(E) \rightarrow \mathbb{D}^{-\infty}(E \otimes H)$  such that (a) each restriction  $D : \mathbb{D}_p^{s+1}(E) \rightarrow \mathbb{D}_p^s(E \otimes H)$  and is continuous for every  $s \in \mathbb{R}$  and  $p \in (1, \infty)$ , and (b) we have  $\langle DF, e \otimes h \rangle_{E \otimes H} = \langle D_h F, e \rangle_E$  for  $e \in E$ ,  $h \in H$  and  $F \in \mathbb{D}_2^1(E)$ , where  $D_h F$  is defined by

$$(2.3) \quad \langle (D_h F)(w), e \rangle_E = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \langle F(w+h) - F(w), e \rangle_E \quad \text{for a.a. } w \in W.$$

The differential operator  $D_h$  in (2.3) is well-defined for a.a.  $w$  because of the so-called *Cameron-Martin theorem*. There also exists a (continuous) linear operator  $D^* : \mathbb{D}^{-\infty}(E \otimes H) \rightarrow \mathbb{D}^{-\infty}(E)$  such that (a)\* each restriction  $D^* : \mathbb{D}_p^{s+1}(E \otimes H) \rightarrow \mathbb{D}_p^s(E)$  and is continuous for every  $s \in \mathbb{R}$  and  $p \in (1, \infty)$ , and (b)\* we have

$$D^*(G \otimes h) = -D_h G + \int_0^T \langle \dot{h}(t), dw(t) \rangle_{\mathbb{R}^d} G$$

for  $h \in H$ ,  $G \in \mathbb{D}_2^1(E)$ . These operators are related as follows: For  $F, G \in \mathbb{D}_2^1(E)$  and  $h \in H$ , it holds that

$$\mathbf{E}[\langle DF, G \otimes h \rangle_{E \otimes H}] = \mathbf{E}[\langle F, D^*(G \otimes h) \rangle_E].$$

Let  $\mathcal{S}(\mathbb{R}^d)$  be the real Schwartz space of rapidly decreasing  $C^\infty$ -functions on  $\mathbb{R}^d$ . We denote by  $\mathcal{S}_{2k}(\mathbb{R}^d)$ ,  $k \in \mathbb{Z}$  the completion of  $\mathcal{S}(\mathbb{R}^d)$  by the norm

$$|\phi|_{2k} := |(1 + x^2 - \Delta/2)^k \phi|_\infty, \quad \phi = \phi(x) \in \mathcal{S}(\mathbb{R}^d),$$

where  $\Delta = \sum_{i=1}^d \partial^2 / (\partial x_i)^2$  and  $|\phi|_\infty = \sup_{x \in \mathbb{R}^d} |\phi(x)|$ .

**Definition 2.1.** (i) A Wiener functional  $F = (F^1, \dots, F^d) \in \mathbb{D}^\infty(\mathbb{R}^d)$  is said to be *non-degenerate* if  $\|\det(\langle DF^i, DF^j \rangle_H)_{i,j}^{-1}\|_p < \infty$  for any  $p \in (1, \infty)$ .  
(ii) A family of Wiener functionals  $F_\alpha = (F_\alpha^1, \dots, F_\alpha^d) \in \mathbb{D}^\infty(\mathbb{R}^d)$ ,  $\alpha \in I$ , where  $I$  is an index set, is said to be *uniformly non-degenerate* if  $\sup_{\alpha \in I} \|\det(\langle DF_\alpha^i, DF_\alpha^j \rangle_H)_{i,j}^{-1}\|_p < \infty$  for any  $p \in (1, \infty)$ .

If  $F \in \mathbb{D}^\infty(\mathbb{R}^d)$  is non-degenerate, then the mapping  $\mathcal{S}(\mathbb{R}^d) \ni \phi \mapsto \phi(F) \in \mathbb{D}^\infty$  extends uniquely to a mapping  $\mathcal{S}'(\mathbb{R}^d) \ni \Lambda \mapsto \Lambda(F) \in \cup_{s>0} \cap_{1<p<\infty} \mathbb{D}_p^{-s}$  such that each restriction maps  $\mathcal{S}_{-2k}(\mathbb{R}^d) \rightarrow \mathbb{D}_p^{-2k}$  and is continuous for every  $k \in \mathbb{Z}$  and  $p \in (1, \infty)$ . The generalized Wiener functional  $\Lambda(F)$  is called the *pull-back* of  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$  by  $F \in \mathbb{D}^\infty(\mathbb{R}^d)$ .

**Lemma 2.2.** Let  $y \in \mathbb{R}$  and  $\delta_y$  be the Dirac delta-function at  $y$ . Then  $\delta_y \in \mathcal{S}_{-2}(\mathbb{R})$ ,  $\delta_y^{(2k)} \in \mathcal{S}_{-2(k+1)}(\mathbb{R})$  for  $k \in \mathbb{N}$ , and  $\sup_{a \in \mathbb{R}} |\delta_a|_{-2} < +\infty$ .

*Proof.* That  $\delta_y \in \mathcal{S}_{-2}(\mathbb{R})$  and  $\delta_y^{(2k)} \in \mathcal{S}_{-2(k+1)}(\mathbb{R})$  for  $k \in \mathbb{N}$  are well known in [11, Chapter V, section 9, Lemma 9.1, p.380]. It is also known that

$$((1 + x^2 - \Delta/2)^{-1} \delta_y)(x) \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)}}{(1 + \frac{\xi^2}{2})} d\xi$$

for any  $x, y \in \mathbb{R}$ , from which, we easily conclude that

$$\sup_{a \in \mathbb{R}} |\delta_a|_{-2} \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi}{(1 + \frac{\xi^2}{2})} < +\infty.$$

□

**2.1. Slight extension to exponential-type distributions.** It will be convenient to extend the pull-back procedure from Schwartz distribution space to the space of all distribution of exponential-type. Let  $\partial_i := \partial/(\partial x_i)$ ,  $i = 1, \dots, d$ .

**Definition 2.3** (Hasumi [10]). We say  $\phi \in C^\infty(\mathbb{R}^d)$  belongs to  $\mathcal{E}(\mathbb{R}^d)$  if  $\sup_{x \in \mathbb{R}^d} |\exp(p|x|) \partial_1^{k_1} \dots \partial_d^{k_d} \phi(x)| < +\infty$  for any  $p \in \mathbb{Z}_{\geq 0}$  and  $k_1, \dots, k_d \in \mathbb{Z}_{\geq 0}$ .

Semi-norms on  $\mathcal{E}(\mathbb{R}^d)$ , defined by

$$|\phi|_p := \sup_{\substack{k_1, \dots, k_d \in \mathbb{Z}_{\geq 0}: \\ 0 \leq k_1 + \dots + k_d \leq p}} \sup_{x \in \mathbb{R}^d} |\exp(p|x|) \partial_1^{k_1} \dots \partial_d^{k_d} \phi(x)|, \quad p = 0, 1, 2, \dots$$

make  $\mathcal{E}(\mathbb{R}^d)$  a locally convex metrizable space and induces continuous inclusions

$$\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{E}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{E}'(\mathbb{R}^d) \hookrightarrow \mathcal{D}'(\mathbb{R}^d)$$

where  $\mathcal{D}'(\mathbb{R}^d)$  is the space of all distributions on  $\mathbb{R}^d$  with the test function space  $\mathcal{D}(\mathbb{R}^d)$ , and  $\mathcal{E}'(\mathbb{R}^d)$  is the continuous dual of  $\mathcal{E}(\mathbb{R}^d)$ . Elements in  $\mathcal{E}'(\mathbb{R}^d)$  are referred as *distributions of exponential-type*. The following is known (see Hasumi [10, Proposition 3]):

**Theorem 2.4.** For any  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$ , there exist  $k \in \mathbb{Z}_{\geq 0}$  and a bounded continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\Lambda = \frac{\partial^{kd}}{\partial x_1^k \dots \partial x_d^k} [\exp(k|x|) f(x)].$$

**Remark 2.1.** The spaces  $\mathcal{E}(\mathbb{R}^d)$  and  $\mathcal{E}'(\mathbb{R}^d)$  are denoted by  $H$  and  $\Lambda_\infty$  respectively in Hasumi [10]. The space  $\mathcal{E}'(\mathbb{R}^d) = \Lambda_\infty$  is firstly introduced by Silva [18] in the study of the Fourier transform of  $\Lambda_\infty$ , called *ultra-distributions* (see Silva [18], Hasumi [10] and Yoshinaga [30]).

Define  $e_k(x) := \prod_{i=1}^d \cosh(kx_i)$  for  $k \in \mathbb{Z}_{\geq 0}$  and  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ .



**Proposition 2.5.** Suppose that  $F_\alpha = (F_\alpha^1, \dots, F_\alpha^d) \in \mathbb{D}^\infty(\mathbb{R}^d)$ ,  $\alpha \in I$ , where  $I$  is an index set, are uniformly non-degenerate and satisfy

$$(2.4) \quad \sup_{\alpha \in I} \mathbf{E}[\exp(r|F_\alpha|_{\mathbb{R}^d})] < +\infty \quad \text{for each } r > 0.$$

Then for any  $p \in (1, \infty)$ ,  $k \in \mathbb{Z}_{\geq 0}$  and  $r > 0$ , there exists a constant  $c = c(q, k, r) > 0$ , where  $1/p + 1/q = 1$ , such that

$$\left\| \frac{\partial^{kd}(e_r \phi)}{\partial x_1^k \dots \partial x_d^k}(F_\alpha) \right\|_{p, -kd} \leq c \sup_{x \in \mathbb{R}^d} |\phi(x)|$$

for all  $\phi \in \mathcal{S}(\mathbb{R}^d)$  and  $\alpha \in I$ .

Now, let  $F \in \mathbb{D}^\infty(\mathbb{R}^d)$  be non-degenerate. Take  $\Lambda = \partial_1^k \dots \partial_d^k [\exp(k|x|)f(x)] \in \mathcal{E}'(\mathbb{R}^d)$  and assume  $k \geq 1$ . Let  $\varepsilon \in (0, 1)$  be arbitrary and put  $r := k + \varepsilon > 0$ . Define  $\phi \in C_0(\mathbb{R}^d)$  (the space of continuous functions on  $\mathbb{R}^d$  vanishing at infinity) by  $\phi(x) := (e^{k|x|} / \prod_{i=1}^d \cosh(rx_i)) f(x)$ , so that now we have  $\Lambda = \partial_1^k \dots \partial_d^k (e_r \phi) = \partial_1^k \dots \partial_d^k [(\prod_{i=1}^d \cosh(rx_i)) \phi(x)]$ . Take any sequence  $\phi_n \in \mathcal{S}(\mathbb{R}^d)$ ,  $n \in \mathbb{N}$  such that  $|\phi_n - \phi|_\infty \rightarrow 0$ . Then Proposition 2.5 tells us that  $\lim_{n \rightarrow \infty} [\partial_1^k \dots \partial_d^k (e_r \phi_n)](F)$  exists in  $\mathbb{D}_p^{-kd}$  for each  $p \in (1, \infty)$ . The limit does not depend on the choice of  $\varepsilon > 0$  and the sequence  $\phi_n \in \mathcal{S}(\mathbb{R}^d)$ . Under these notations,

**Definition 2.6.** We denote the limit by  $\Lambda(F)$  and call the *pull-back* of  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$  by  $F$ .

*Proof of Proposition 2.5.* Let  $p \in (1, \infty)$ ,  $k \in \mathbb{Z}_{\geq 0}$ ,  $r > 0$ ,  $\alpha \in I$ ,  $\phi \in \mathcal{S}(\mathbb{R}^d)$  and  $J \in \mathbb{D}^\infty$  be arbitrary. Then

$$\mathbf{E}[(\partial_1^k \dots \partial_d^k (e_r \phi))(F_\alpha)J] = \mathbf{E}\left[\left(\prod_{i=1}^d \cosh(rF_\alpha^i)\right)\phi(F_\alpha)l_\alpha(J)\right]$$

where  $l_\alpha(J) \in \mathbb{D}^\infty$  is of the form

$$l_\alpha(J) = \sum_{j=0}^{kd} \langle P_j(w), D^j J \rangle_{H^{\otimes j}}$$

for some  $P_j(w) \in \mathbb{D}^\infty(H^{\otimes j})$ ,  $j = 0, 1, \dots, kd$  which are polynomials in  $F_\alpha$ , its derivatives up to the order  $kd$ , and  $\det(\langle DF_\alpha^i, DF_\alpha^j \rangle_H)_{ij}^{-1}$ . Take  $q' \in (1, q)$ . Since  $\{F_\alpha\}_{\alpha \in I}$  is uniformly non-degenerate, there exists  $c_0 > 0$  such that

$$\|l_\alpha(J)\|_{q'} \leq c_0 \|J\|_{q, k} \quad \text{for all } \alpha \in I \text{ and } J \in \mathbb{D}^\infty.$$

Therefore by taking  $p' \in (1, \infty)$  such that  $1/p' + 1/q' = 1$ , we have

$$|\mathbf{E}[\left(\prod_{i=1}^d \cosh(rF_\alpha^i)\right)\phi(F_\alpha)l_\alpha(J)]| \leq c'_0 |\phi|_\infty \|\exp(r|F_\alpha|)\|_{p'} \|J\|_{q, k},$$



for some constant  $c'_0 > 0$ , which implies

$$\|(e_k \phi)^{(k)}(F_\alpha)\|_{p,-k} \leq c'_0 |\phi|_\infty \|\exp(r|F_\alpha|)\|_{p'} \leq c'_0 \sup_{\alpha \in I} \|\exp(r|F_\alpha|)\|_{p'} |\phi|_\infty.$$

□

**Corollary 2.7.** *Suppose that  $F_\alpha \in \mathbb{D}^\infty(\mathbb{R}^d)$ ,  $\alpha \in I \subset \mathbb{R}$ , where  $I$  is an index set, satisfy*

- (i)  $\{F_\alpha\}_{\alpha \in I}$  is uniformly non-degenerate;
- (ii)  $\sup_{\alpha \in I} \mathbf{E}[\exp(r|F_\alpha|_{\mathbb{R}^d})] < +\infty$  for each  $r > 0$ ;
- (iii) the mapping  $I \ni \alpha \mapsto F_\alpha \in \mathbb{D}^\infty(\mathbb{R}^d)$  is continuous.

Then for any  $p \in (1, \infty)$  and  $\Lambda = \partial_1^k \cdots \partial_d^k [\exp(k|x|)f(x)] \in \mathcal{E}'(\mathbb{R}^d)$ , the mapping

$$I \ni \alpha \mapsto \Lambda(F_\alpha) \in \mathbb{D}_p^{-kd}$$

is continuous.

*Proof.* Let  $p \in (1, \infty)$ ,  $\alpha \in I$  and  $\varepsilon > 0$  be arbitrary. Suppose that  $\Lambda = \partial_1^k \cdots \partial_d^k [(\prod_{i=1}^d \cosh(rx_i))\phi(x)]$  where  $r > k \geq 1$  and  $\phi \in C_0(\mathbb{R}^d)$ . Then by Proposition 2.5, there exists  $\psi \in \mathcal{S}(\mathbb{R}^d)$  such that  $\|\Lambda(F_\beta) - (e_r \psi)^{(k)}(F_\beta)\|_{p,-k} < \varepsilon/3$  for every  $\beta \in I$ . Furthermore, since by the condition (iii), there exists  $\delta > 0$  such that  $\|(e_r \psi)^{(k)}(F_\beta) - (e_r \psi)^{(k)}(F_\alpha)\|_p < \varepsilon/3$  if  $|\beta - \alpha| < \delta$ . Hence, for each  $\beta \in I$  with  $|\beta - \alpha| < \delta$ ,

$$\begin{aligned} & \|\Lambda(F_\alpha) - \Lambda(F_\beta)\|_{p,-k} \\ & \leq \|\Lambda(F_\alpha) - (e_r \psi)^{(k)}(F_\alpha)\|_{p,-k} \\ & \quad + \|(e_r \psi)^{(k)}(F_\alpha) - (e_r \psi)^{(k)}(F_\beta)\|_p \\ & \quad + \|(e_r \psi)^{(k)}(F_\beta) - \Lambda(F_\beta)\|_{p,-k} \\ & < \varepsilon. \end{aligned}$$

The case of  $k = 0$  is clear. □

### 3. BOCHNER INTEGRABILITY OF PULL-BACKS BY DIFFUSIONS

Let  $w = (w(t))_{t \geq 0}$  be a  $d$ -dimensional Wiener process with  $w(0) = 0$ . Let  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ ,  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . We consider the following stochastic differential equation

$$(3.1) \quad dX_t = \sigma(X_t)dw(t) + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d.$$

In this section, we assume conditions (H1) and (H2). Under these conditions, the equation (3.1) admits a unique strong solution. We denote by  $\{X(t, x, w)\}_{t \geq 0}$  the unique strong solution  $X = (X_t)_{t \geq 0}$  to (3.1). Furthermore, for each  $t > 0$  and  $x \in \mathbb{R}$ , we have  $X(t, x, w) \in \mathbb{D}^\infty(\mathbb{R}^d)$  and is non-degenerate. Henceforward for  $t > 0$ , one can define the pull-back  $\Lambda(X(t, x, w))$  of  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$  by  $X(t, x, w)$  as an element of  $\cup_{s>0} \cap_{1<p<\infty} \mathbb{D}_p^{-s}$ .

Fix  $T > 0$  be arbitrary. By the condition (H2), we further know that

$$\sup_{t \in K} \|\det(\langle DX^i(t, x, w), DX^j(t, x, w) \rangle_H)_{ij}^{-1}\|_p < +\infty \quad \text{for } 1 < p < \infty$$

holds for each closed interval  $K \subset (0, T]$ , which implies that for each  $k \in \mathbb{Z}_{\geq 0}$ ,  $p \in (1, \infty)$  and  $\Lambda \in \mathcal{S}_{-2k}(\mathbb{R}^d)$ , the mapping  $(0, T] \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_p^{-2k}$  is continuous (see e.g. [26, Remark 2.2]). In particular, the mapping  $[t_0, T] \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_p^{-2k}$  is Bochner integrable for each  $t_0 > 0$ , and hence the Bochner integral  $\int_{t_0}^T \Lambda(X(t, x, w)) dt$  makes sense as an element in  $\mathbb{D}_p^{-2k}$  for each  $t_0 > 0$ . Note that analogous results follow also for  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$  by virtue of Corollary 2.7.

Now our problem is the Bochner integrability on  $(0, T]$ , i.e., whether  $\int_0^T \|\Lambda(X(t, x, w))\|_{p, -2k} dt < +\infty$  holds or not. The main results of this section is Theorem 3.3 and Proposition 3.10. Before seeing this, we shall start with the Brownian case, which would be an introductory example.

### 3.1. Bochner integrability of $\delta_0(w(t))$ .

**Proposition 3.1.** *Let  $d = 1$ ,  $\Lambda \in \mathcal{S}'(\mathbb{R})$  and  $s \in \mathbb{R}$ . If  $\Lambda(w(T)) \in \mathbb{D}_2^s$  then the mapping*

$$(0, T] \ni t \mapsto \sqrt{\frac{T}{t}} \Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) \in \mathbb{D}_2^s$$

*is Bochner integrable in  $\mathbb{D}_2^s$  and we have*

$$\int_0^T \sqrt{\frac{T}{t}} \Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) dt \in \mathbb{D}_2^{s+1}.$$

Hence the Bochner integral poses a sort of “smoothing effect”, in the sense of raising the regularity-index  $s$ , which might be a common understanding for most of us.

*Proof.* Recall the integration by parts formula

$$\mathbf{E}[\Lambda'(w(t)) H_n\left(\frac{w(t)}{\sqrt{t}}\right)] = t^{-1/2} \mathbf{E}[\Lambda(w(t)) H_{n+1}\left(\frac{w(t)}{\sqrt{t}}\right)]$$

where  $H_n := \partial^{*n} 1$  is the  $n$ -th Hermite polynomial and  $\partial^* = -\partial + x$ . The family  $\{\frac{1}{\sqrt{n!}} H_n\}_{n=0}^\infty$  forms a CONB of  $L_2(\mathbb{R}, (2\pi)^{-1/2} e^{-x^2/2} dx)$  (see Lemma A.1 in Appendix A).

Then  $\Lambda((T/t)^{1/2} w(t))$  has the Fourier expansion (the Wiener-Itô chaos expansion)

$$\Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{E}[\Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) H_n\left(\frac{w(t)}{\sqrt{t}}\right)] H_n\left(\frac{w(t)}{\sqrt{t}}\right),$$

and hence

$$\|\Lambda\left(\sqrt{\frac{T}{t}} w(t)\right)\|_{2,s}^2 = \sum_{n=0}^{\infty} \frac{(1+n)^s}{n!} \mathbf{E}[\Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) H_n\left(\frac{w(t)}{\sqrt{t}}\right)]^2$$

Here we have

$$\mathbf{E}[\Lambda(\sqrt{\frac{T}{t}}w(t))H_n(\frac{w(t)}{\sqrt{t}})] = \mathbf{E}[\Lambda(w(T))H_n(\frac{w(T)}{\sqrt{T}})].$$

Therefore

$$\begin{aligned} & \|\sqrt{\frac{T}{t}}\Lambda(\sqrt{\frac{T}{t}}w(t))\|_{2,s}^2 \\ &= \frac{T}{t} \sum_{n=0}^{\infty} \frac{(1+n)^s}{n!} \mathbf{E}[\Lambda(w(T))H_n(\frac{w(T)}{\sqrt{T}})]^2 = \frac{T}{t} \|\Lambda(w(T))\|_{2,s}^2, \end{aligned}$$

that is, we get  $\|(T/t)^{1/2}\Lambda((T/t)^{1/2}w(t))\|_{2,s} = (T/t)^{1/2}\|\Lambda(w(T))\|_{2,s}$ . Since

$$\int_0^T \|\sqrt{\frac{T}{t}}\Lambda(\sqrt{\frac{T}{t}}w(t))\|_{2,s} dt = T^{1/2} \|\Lambda(w(T))\|_{2,s} \int_0^T t^{-1/2} dt < +\infty,$$

the function  $(0, T] \ni t \mapsto (T/t)^{1/2}\Lambda((T/t)^{1/2}w(t)) \in \mathbb{D}_2^s$  is Bochner integrable and  $\int_0^T (T/t)^{1/2}\Lambda((T/t)^{1/2}w(t)) dt \in \mathbb{D}_2^s$ .

Next we show that in fact we have

$$(3.2) \quad \int_0^T \sqrt{\frac{T}{t}}\Lambda(\sqrt{\frac{T}{t}}w(t)) dt \in \mathbb{D}_2^{s+1}.$$

For this, we note that

$$\begin{aligned} & \int_0^T \sqrt{\frac{T}{t}}\Lambda(\sqrt{\frac{T}{t}}w(t)) dt \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{E}[\Lambda(w(T))H_n(\frac{w(T)}{\sqrt{T}})] \int_0^T \sqrt{\frac{T}{t}}H_n(\frac{w(t)}{\sqrt{t}}) dt \end{aligned}$$

is the chaos expansion for  $\int_0^T (T/t)^{1/2}\Lambda((T/t)^{1/2}w(t)) dt$  (more precisely, we have used Corollary 3.4 below). We shall focus on the  $L_2$ -norm of the last factor:

$$\begin{aligned} & \mathbf{E}[\{\int_0^T \frac{1}{\sqrt{t}}H_n(\frac{w(t)}{\sqrt{t}}) dt\}^2] \\ &= \int_{0 < t < s < T} \frac{1}{\sqrt{ts}} \mathbf{E}[H_n(\frac{w(t)}{\sqrt{t}})H_n(\frac{w(s)}{\sqrt{s}})] dt ds \\ &= \int_{0 < t < s < T} \frac{1}{\sqrt{ts}} \mathbf{E}[H_n(\frac{w(t)}{\sqrt{t}})H_n(\sqrt{\frac{t}{s}}\frac{w(s)}{\sqrt{t}} + \frac{w(s) - w(t)}{\sqrt{s}})] dt ds \\ &= \int_{0 < t < s < T} \frac{1}{\sqrt{ts}} n! \left(\frac{t}{s}\right)^{n/2} dt ds \\ &= n! \int_{0 < t < s < T} t^{-(n-1)/2} s^{-(n+1)/2} dt ds = \frac{4T}{n+1} n!. \end{aligned}$$

Hence we have

$$\begin{aligned}
& \left\| \int_0^T \sqrt{\frac{T}{t}} \Lambda\left(\sqrt{\frac{T}{t}} w(t)\right) dt \right\|_{2,s+1}^2 \\
&= T \sum_{n=0}^{\infty} \frac{(1+n)^{s+1}}{(n!)^2} \mathbf{E}[\Lambda(w(T)) H_n\left(\frac{w(T)}{\sqrt{T}}\right)]^2 \mathbf{E}\left[\left\{ \int_0^T \frac{1}{\sqrt{t}} H_n\left(\frac{w(t)}{\sqrt{t}}\right) dt \right\}^2\right] \\
&= 4T^2 \sum_{n=0}^{\infty} (1+n)^s \mathbf{E}[\Lambda(w(T)) H_n\left(\frac{w(T)}{\sqrt{T}}\right)]^2 = 4T^2 \|\Lambda(w(T))\|_{2,s}^2 < +\infty,
\end{aligned}$$

which proves (3.2).  $\square$

By Nualart-Vives [16, Section 2] and Watanabe [28], it is known that

$$\delta_0(w(t)) \in \mathbb{D}_2^{(-1/2)-} \quad \text{but} \quad \delta_0(w(t)) \notin \mathbb{D}_2^{-1/2} \quad \text{for } t > 0,$$

where  $\mathbb{D}_2^{s-} := \cap_{\alpha < s} \mathbb{D}_2^\alpha$ . From this fact and Proposition 3.1, we reached the following result by Nualart-Vives [16] and Watanabe [28].

**Corollary 3.2.** *If  $d = 1$ , we have  $\int_0^T \delta_0(w(t)) dt \in \mathbb{D}_2^{(1/2)-}$ .*

### 3.2. Bochner integrability of $\Lambda(X_t)$ where $\Lambda$ is a distribution.

A distribution  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$  is said to be *positive* if  $\langle \Lambda, f \rangle \geq 0$  for all nonnegative  $f \in \mathcal{S}(\mathbb{R}^d)$ . Then it is known that there exists a Radon measure  $\mu$  on  $\mathbb{R}^d$  such that

$$\langle \Lambda, f \rangle = \int_{\mathbb{R}^d} \langle \delta_y, f \rangle \mu(dy), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The main objective in this subsection is to prove

**Theorem 3.3.** *Let  $d = 1$  and  $x \in \mathbb{R}$ . Suppose (H1),  $\sigma(x)^2 > 0$  and that  $\Lambda \in \mathcal{S}'(\mathbb{R})$  be positive. Then for every  $p \in (1, \infty)$ , we have*

$$(3.3) \quad \int_0^T \int_{\mathbb{R}} \|\delta_y(X(t, x, w))\|_{p,-2} \mu(dy) dt < +\infty.$$

**Remark 3.1.** From this, if  $\Lambda \in \mathcal{S}'(\mathbb{R})$  is positive, then we see that  $\int_0^T \|\Lambda(X(t, x, w))\|_{p,-2} dt < +\infty$ , and hence  $(0, T] \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_p^{-2}$  is Bochner integrable. Furthermore,  $(0, T] \times \mathbb{R} \ni (t, y) \mapsto \delta_y(X(t, x, w)) \in \mathbb{D}_p^{-2}$  is also Bochner integrable with respect to  $dt \otimes \mu(dy)$ , and we have

$$\begin{aligned}
\int_0^T \Lambda(X(t, x, w)) dt &= \int_0^T \int_{\mathbb{R}} \delta_y(X(t, x, w)) \mu(dy) dt \\
&= \int_{\mathbb{R}} \int_0^T \delta_y(X(t, x, w)) dt \mu(dy) \quad \text{in } \mathbb{D}_p^{-2}.
\end{aligned}$$

**Corollary 3.4.** *Let  $d = 1$ . For each  $y \in \mathbb{R}$  and  $n \in \mathbb{Z}_{\geq 0}$ , the mapping  $(0, T] \ni t \mapsto J_n[\delta_y(X_t)] \in L_2$  is Bochner integrable and*

$$J_n\left[\int_0^T \delta_y(X_t) dt\right] = \int_0^T J_n[\delta_y(X_t)] dt.$$

*Proof of Corollary 3.4.* By Theorem 3.3, we have

$$\begin{aligned} \int_0^T \|J_n[\delta_y(X_t)]\|_{L_2} dt &= (1+n) \int_0^T \|J_n[\delta_y(X_t)]\|_{2,-2} dt \\ &\leq (1+n) \int_0^T \|\delta_y(X_t)\|_{2,-2} dt < +\infty. \end{aligned}$$

This shows the Bochner integrability of the mapping  $(0, T] \ni t \mapsto J_n[\delta_y(X_t)] \in L_2$ , and hence  $\int_0^T J_n[\delta_y(X_t)] dt \in L_2$ .

Secondly, for each  $F \in \mathbb{D}^\infty$ , the mapping  $\mathbb{D}_2^s \ni G \mapsto \mathbf{E}[GF] \in \mathbb{R}$  is linear and bounded for each  $s \in \mathbb{R}$ . Therefore Bochner integrals and (generalized) expectations may be interchanged, and accordingly we have

$$\begin{aligned} \mathbf{E}\left[J_n\left[\int_0^T \delta_y(X_t) dt\right]F\right] &= \mathbf{E}\left[\int_0^T \delta_y(X_t) dt J_n F\right] \\ &= \int_0^T \mathbf{E}[\delta_y(X_t) J_n F] dt = \int_0^T \mathbf{E}[\delta_y(X_t) J_n F] dt \\ &= \int_0^T \mathbf{E}[J_n[\delta_y(X_t)]F] dt = \mathbf{E}\left[\int_0^T J_n[\delta_y(X_t)] dt F\right], \end{aligned}$$

which proves the second assertion.  $\square$

To prove Theorem 3.3, we need several implements.

For each  $\varepsilon > 0$ , we consider the following  $d$ -dimensional stochastic differential equation

$$(3.4) \quad dX_t = \varepsilon \sigma(X_t) dw(t) + \varepsilon^2 b(X_t) dt.$$

Similarly to the equation (3.1), we denote by  $\{X^\varepsilon(t, x, w)\}_{t \geq 0}$  a unique strong solution  $X^\varepsilon = (X_t^\varepsilon)_{t \geq 0}$  to (3.4) such that  $X_0^\varepsilon = x \in \mathbb{R}^d$ . Then it holds that for each  $\varepsilon > 0$ ,  $\{X(\varepsilon^2 t, x, w)\}_{t \geq 0}$  is equivalent to  $\{X^\varepsilon(t, x, w)\}_{t \geq 0}$  in law. A more tricky fact which we need is the following.

**Proposition 3.5.** *Let  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ . Then for every  $p \in (1, \infty)$ ,  $s \in \mathbb{R}$  and  $t > 0$ , we have  $\|\Lambda(X(\varepsilon^2 t, x, w))\|_{p,s} = \|\Lambda(X^\varepsilon(t, x, w))\|_{p,s}$ .*

*Proof.* For simplicity, we give a proof in the case of  $d = 1$ . Let  $p \in (1, \infty)$ ,  $s \in \mathbb{R}$  and  $t > 0$  be arbitrary. It is enough to prove that  $\|f(X(\varepsilon^2 t, x, w))\|_{p,s} = \|f(X^\varepsilon(t, x, w))\|_{p,s}$  for each  $f \in \mathcal{S}(\mathbb{R})$ . By the Veretennikov-Krylov formula (see [24, p.251, Theorem 4]), we have

$$\begin{aligned} &J_n[f(X^\varepsilon(t, x, w))] \\ &= \int_0^T \cdots \int_0^{r_2} (P_{r_1}^\varepsilon Q_{r_2-r_1}^\varepsilon \cdots Q_{r_n-r_{n-1}}^\varepsilon Q_{t-r_n}^\varepsilon f)(x) dw(r_1) \cdots dw(r_n), \end{aligned}$$

where  $(P_r^\varepsilon f)(z) := \mathbf{E}[f(X^\varepsilon(r, z, w))]$  for  $z \in \mathbb{R}$ ,  $Q_r^\varepsilon f := A^\varepsilon(P_r^\varepsilon f)$  and  $A^\varepsilon := \varepsilon \sigma \frac{d}{dx}$ . In the case  $\varepsilon = 1$ , we will write  $P_r := P_r^1$ ,  $A := A^1$  and

$Q_r := Q_r^1$  for simplicity. Therefore, we have to show that

$$\begin{aligned} & \left\{ \int_0^{\varepsilon^2 t} \cdots \int_0^{\varepsilon^2 r_2} (P_{r_1} Q_{r_2-r_1} \cdots Q_{r_n-r_{n-1}} Q_{t-r_n} f)(x) dw(r_1) \cdots dw(r_k) \right\}_{k=0}^n \\ &= \left\{ \int_0^t \cdots \int_0^{r_2} (P_{r_1}^\varepsilon Q_{r_2-r_1}^\varepsilon \cdots Q_{r_n-r_{n-1}}^\varepsilon Q_{t-r_n}^\varepsilon f)(x) dw(r_1) \cdots dw(r_k) \right\}_{k=0}^n \end{aligned}$$

in law for each  $n \in \mathbb{N}$ , and hence by the scaling property of Brownian motion:  $(\varepsilon^{-1}w(\varepsilon^2 t))_{t \geq 0} = (w(t))_{t \geq 0}$  in law, it suffices to show that

$$\begin{aligned} (3.5) \quad & (P_{\varepsilon^2 r_1} Q_{\varepsilon^2 r_2 - \varepsilon^2 r_1} \cdots Q_{\varepsilon^2 r_n - \varepsilon^2 r_{n-1}} Q_{\varepsilon^2 t - \varepsilon^2 r_n} f)(x) \\ &= \varepsilon^{-n} (P_{r_1}^\varepsilon Q_{r_2-r_1}^\varepsilon \cdots Q_{r_n-r_{n-1}}^\varepsilon Q_{t-r_n}^\varepsilon f)(x) \end{aligned}$$

for each  $0 < r_1 < \cdots < r_n < t$ . But this is clear since we have  $P_{\varepsilon^2 r} = P_r^\varepsilon$ ,  $A = \varepsilon^{-1}A^\varepsilon$  and  $Q_{\varepsilon^2 r} = \varepsilon^{-1}Q_r^\varepsilon$  for each  $r > 0$ .  $\square$

For each  $\varepsilon > 0$  and  $x \in \mathbb{R}^d$ , we set

$$F(\varepsilon, x, w) := \frac{X^\varepsilon(1, x, w) - x}{\varepsilon}.$$

Then the following two conditions are equivalent (see [26, Theorem 3.4]):

- (H2), i.e., there exists  $\lambda > 0$  such that

$$\lambda |\xi|^2 \leq \langle \xi, (\sigma^* \sigma)(x) \xi \rangle_{\mathbb{R}^d}, \quad \text{for all } \xi \in \mathbb{R}^d.$$

- the family  $\{F(\varepsilon, x, w)\}_{\varepsilon > 0}$  is uniformly non-degenerate.

**Proposition 3.6.** *Let  $d = 1$  and  $x \in \mathbb{R}$ . Suppose (H1) and  $\sigma(x)^2 > 0$ . Then for any  $p \in (1, \infty)$ , we have  $\limsup_{0 < \varepsilon \leq 1} \|\delta_0(F(\varepsilon, x, w))\|_{p, -2} < +\infty$ .*

*Proof.* Let  $\phi(z) := (1 + z^2 - \Delta)^{-1} \delta_0(z) \in \mathcal{S}_0$  and take  $q \in (1, \infty)$  so that  $1/p + 1/q = 1$ . Then for each  $J \in \mathbb{D}^\infty$ , we have

$$\begin{aligned} \mathbf{E}[\delta_0(F(\varepsilon, x, w))J] &= \mathbf{E}[(1 + z^2 - \Delta)\phi](F(\varepsilon, x, w))J \\ &= \mathbf{E}[\phi(F(\varepsilon, x, w))l_\varepsilon(J)] \end{aligned}$$

where  $l_\varepsilon(J) \in \mathbb{D}^\infty$  is of the form

$$l_\varepsilon(J) = \langle P_0(\varepsilon, w), DJ \rangle_H + \langle P_1(\varepsilon, w), D^2 J \rangle_{H \otimes H}$$

for some  $P_i(\varepsilon, w) \in \mathbb{D}^\infty(H^{\otimes i})$ ,  $i = 1, 2$ , both of which are polynomials in  $F(\varepsilon, x, w)$ , its derivatives up to the second order and  $\|DX^\varepsilon(1, x, w)\|_H^{-2}$ .

Take  $q' \in (1, q)$ . Since  $\{F(\varepsilon, x, w)\}_{\varepsilon > 0}$  is uniformly non-degenerate, there exists  $c_0 > 0$  such that

$$\|l_\varepsilon(J)\|_{q'} \leq c_0 \|J\|_{q, 2} \quad \text{for all } \varepsilon \in (0, 1] \text{ and } J \in \mathbb{D}^\infty.$$

Therefore we have for each  $J \in \mathbb{D}^\infty$ ,

$$|\mathbf{E}[\delta_0(F(\varepsilon, x, w))J]| \leq \|\phi\|_\infty \|l_\varepsilon(J)\|_{q'} \leq c_0 \|\phi\|_\infty \|J\|_{q, 2}$$

which implies  $\sup_{0 < \varepsilon \leq 1} \|\delta_0(F(\varepsilon, x, w))\|_{p, -2} \leq c_0 \|\phi\|_\infty < +\infty$ .  $\square$

Secondly, we recall the next fact (see Ikeda-Watanabe [11, Chapter V, section 8, p.384]) to prove Theorem 3.3.

**Proposition 3.7.** *Let  $x \in \mathbb{R}^d$  and suppose that (H1) and (H2). Then for any  $p \in (1, \infty)$  and  $k \in \mathbb{Z}_{\geq 0}$ , there exists  $C > 0$  such that*

$$\|\phi \circ F(\varepsilon, x, w)\|_{p, -2k} \leq C \|\phi\|_{-2k} \quad \text{for } \phi \in \mathcal{S}(\mathbb{R}^d), \varepsilon \in (0, T].$$

The last tool we need is the following.

**Lemma 3.8.** *Let  $x \in \mathbb{R}^d$  and suppose that (H1) and (H2). Then for each  $k_1, \dots, k_d \in \mathbb{Z}_{\geq 0}$ ,  $K > |x|$  and  $p \in (1, \infty)$ , there exist constants  $\nu_0, c_1, c_2 > 0$  such that*

$$\|\partial_1^{k_1} \dots \partial_d^{k_d} \delta_y(X(t, x, w))\|_{p, -(n+2)} \leq c_1 t^{-\nu_0} \exp \left\{ -c_2 \frac{|x-y|^2}{t} \right\}$$

for each  $t \in (0, T]$  and  $y \in \mathbb{R}$  such that  $|y| \geq K$ , where  $n := k_1 + \dots + k_d$ .

*Proof.* For simplicity of notation, we prove the case  $d = 1$ . It suffices to prove that for each  $n \in \mathbb{Z}_{\geq 0}$ ,  $K > |x|$  and  $p \in (1, \infty)$ , there exist constants  $\nu_0, c_1, c_2 > 0$  such that

$$|\mathbf{E}[J \delta_y^{(n)}(X_t)]| \leq c_1 t^{-\nu_0} \exp \left\{ -c_2 \frac{|x-y|^2}{t} \right\} \|J\|_{p, n+2}, \quad J \in \mathbb{D}^\infty$$

for each  $t \in (0, T]$  and  $y \in \mathbb{R}$  such that  $|y| \geq K$ .

Let  $y \in \mathbb{R}$  be arbitrary and  $\varphi(z) := (1 + z^2 - \Delta)^{-1} \delta_y(z) \in \mathcal{S}_0$ . Take a  $C^\infty$ -function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\phi(\xi) = \begin{cases} 1 & \text{if } \xi \leq 1/3, \\ 0 & \text{if } \xi \geq 2/3 \end{cases}$$

and then we set  $\psi(z) := \phi\left(\frac{z-y}{|x-y|}\right)$ . Let  $p \in (1, \infty)$  be arbitrary and let  $q \in (1, \infty)$  be such that  $1/p + 1/q = 1$ . Further take  $q' \in (1, q)$ . Since  $\psi \delta_y = \delta_y$ , we have for each  $J \in \mathbb{D}^\infty$  that

$$\begin{aligned} \mathbf{E}[J \delta_y^{(n)}(X_t)] &= \mathbf{E}[J \psi(X_t) \left( \frac{d^n}{dz^n} (1 + z^2 - \Delta) \varphi \right)(X_t)] \\ (3.6) \quad &= \mathbf{E}[\varphi(X_t) \left\{ \sum_{j=0}^{n+2} \psi^{(j)}(X_t) l_{j,t}(J) \right\}], \end{aligned}$$

where  $\psi^{(j)}$  is the  $j$ -th derivative of  $\psi$  (with convention that  $\psi^{(0)} = \psi$ ) and each  $l_{j,t}(J)$  is of the form

$$(3.7) \quad P_0(t, w)J + \langle P_1(t, w), DJ \rangle_H + \dots + \langle P_{n+2}(t, w), D^{n+2}J \rangle_{H^{\otimes(n+2)}}$$

for some  $P_i(t, w) \in \mathbb{D}^\infty(H^{\otimes i})$ ,  $i = 0, 1, \dots, n+2$ , which is a polynomial in  $X_t = X(t, x, w)$ , its derivatives and  $\|DX(t, x, w)\|_H^{-2}$ , but does not depend on  $\psi$ .



Note that  $\psi(z) \equiv 0$  for  $|z - x| < \frac{|y-x|}{3}$ , and hence the last equation in (3.6) equals to

$$\mathbf{E}\left[\left(\sum_{j=0}^{n+2}(\varphi\psi^{(j)})(X_t)l_{j,t}(J)\right)1_{\{|X_t - x| \geq \frac{|y-x|}{3}\}}\right].$$

Therefore, by taking  $p' \in (1, \infty)$  such that  $1/p' + 1/q' = 1$ , we have

$$(3.8) \quad |\mathbf{E}[J\delta_y(X_t)]| \leq \sum_{j=0}^{n+2} \|\varphi\psi^{(j)}\|_{\infty} \|l_{j,t}(J)\|_{q'} \mathbf{P}\left(|X_t - x| \geq \frac{|y-x|}{3}\right)^{1/p'}.$$

Henceforth we shall focus on each factor in the last equation.

Firstly, note that  $\psi$  has a parameter  $y$ . Since  $\psi^{(j)}(z) = |x-y|^{-j} \phi^{(j)}((z-y)/|x-y|)$  for  $j = 0, 1, \dots, n+2$ , and by Lemma 2.2 we have

$$(3.9) \quad \sup_{\substack{y \in \mathbb{R} \\ |y| \geq K}} \|\varphi\psi^{(j)}\|_{\infty} < +\infty \quad \text{for } j = 0, 1, \dots, n+2.$$

Secondly, it is well known that for each  $r \in (1, \infty)$ , there exist  $\nu, K_1(r) > 0$  such that

$$\| \|DX(t, x, w)\|_H^{-2} \|_r \leq K_1(r) t^{-\nu} \quad \text{for any } t \in (0, T]$$

(see [14, (3.25) Corollary p.22] or [11, Chapter V, section 10, Theorem 10.2]). Now, bearing in mind the form (3.7), we see that there exist  $\nu_0, c > 0$  such that

$$(3.10) \quad \|l_{j,t}(J)\|_{q'} \leq ct^{-\nu_0} \|J\|_{p,2} \quad \text{for } j = 0, 1, \dots, n+2 \text{ and } t \in (0, T].$$

Thirdly, we have also that there exist  $a_1, a_2 > 0$  such that

$$(3.11) \quad \mathbf{P}\left(|X(t, x, w) - x| \geq \frac{|y-x|}{3}\right) \leq a_1 \exp\left\{-a_2 \frac{|x-y|^2}{t}\right\}$$

for all  $t \in (0, T]$  and  $x, y \in \mathbb{R}$  (see [11, Chapter V, section 10, Lemma 10.5]).

Now, combining (3.8), (3.9), (3.10) and (3.11), we obtain the result.  $\square$

We are now in a position to prove Theorem 3.3.

*Proof of Theorem 3.3.* We begin with the following:

$$\frac{|x-y|^2}{y^2} \geq \frac{(|x|-|y|)^2}{y^2} = \frac{y^2 - 2|xy| + x^2}{y^2} \rightarrow 1 \quad \text{as } |y| \rightarrow +\infty,$$

which implies that there exists  $K' > 0$  such that

$$(3.12) \quad |x-y|^2 \geq \frac{y^2}{2}, \quad \text{for any } |y| > K'.$$

Let  $K := \max\{K', |x| + 1\}$ . Then we divide the integral (3.3) as

$$\begin{aligned} & \int_0^T \int_{|y|>K} \|\delta_y(X(t, x, w))\|_{p,-2} \mu(dy) dt \\ & + \int_0^T \int_{|y|\leq K} \|\delta_y(X(t, x, w))\|_{p,-2} \mu(dy) dt =: I_1 + I_2 \quad (\text{say}), \end{aligned}$$

respectively.

We shall look at the integral  $I_1$ . By Proposition 3.5, Lemma 3.8 and (3.12), there exist  $\nu_0, c_1, c_2 > 0$  such that we have

$$\|\delta_y(X(t, x, w))\|_{p,-2} \leq c_1 t^{-\nu_0} \exp\left\{-c_2 \frac{|x-y|^2}{t}\right\} \leq c_1 t^{-\nu_0} \exp\left\{-c_2 \frac{|y|^2}{2t}\right\} \quad (3.13)$$

for  $|y| > K$  and  $t \in (0, T]$ . To dominate the last quantity, we shall prove that for some  $c_3 > 0$ , it holds that

$$t^{-\nu_0} \exp\left\{-c_2 \frac{|y|^2}{2t}\right\} \leq c_3 \exp\left\{-c_2 \frac{|y|^2}{4t}\right\} \quad \text{for } t \in (0, T] \text{ and } |y| > K. \quad (3.14)$$

Indeed, we have

$$\frac{t^{-\nu_0} \exp\left\{-c_2 \frac{|y|^2}{2t}\right\}}{\exp\left\{-c_2 \frac{|y|^2}{4t}\right\}} = t^{-\nu_0} \exp\left\{-c_2 \frac{|y|^2}{4t}\right\} \leq t^{-\nu_0} \exp\left\{-c_2 \frac{K^2}{4t}\right\} \rightarrow 0$$

as  $t \downarrow 0$ , which proves (3.14). Combining (3.13) and (3.14), we obtain

$$\begin{aligned} I_1 & \leq c_1 c_3 \int_0^T \int_{|y|>K} \exp\left\{-c_2 \frac{|y|^2}{2t}\right\} \mu(dy) dt \\ & \leq c_1 c_3 T \int_{-\infty}^{+\infty} \exp\left\{-c_2 \frac{|y|^2}{2T}\right\} \mu(dy) = c_1 c_3 T \langle \Lambda, f \rangle < +\infty \end{aligned}$$

where  $f \in \mathcal{S}(\mathbb{R})$  is given by  $f(y) := e^{-c_2 y^2/(2T)}$ ,  $y \in \mathbb{R}$ .

Next we turn to  $I_2$ . By Proposition 3.7 and with noting that  $\sup_{a \in \mathbb{R}} \|\delta_a\|_{-2} < +\infty$  (Proposition 2.2), we have

$$\begin{aligned} & t^{1/2} \sup_{y \in \mathbb{R}} \|\delta_y(X(t, x, w))\|_{p,-2} = t^{1/2} \sup_{y \in \mathbb{R}} \|\delta_y(X^{\sqrt{t}}(1, x, w))\|_{p,-2} \\ & = \sup_{y \in \mathbb{R}} \|\delta_{(x-y)/\sqrt{t}}(F(\sqrt{t}, x, w))\|_{p,-2} \leq C \sup_{a \in \mathbb{R}} \|\delta_a\|_{-2} \end{aligned}$$

for each  $t > 0$ . Hence we can conclude that there exists  $C' > 0$  such that

$$\|\delta_y(X(t, x, w))\|_{p,-2} \leq C' t^{-1/2} \quad \text{for } |y| \leq K, t \in (0, T].$$

Now, it is easy to deduce that

$$I_2 \leq C' \mu(\{y \in \mathbb{R} : |y| \leq K\}) \int_0^T t^{-1/2} dt < +\infty.$$

□

Recall that the support of  $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$  is defined as the complement of the set of all  $y \in \mathbb{R}^d$  for which there exists an open neighbourhood  $U$  such that for any  $f \in \mathcal{D}(\mathbb{R}^d)$  with  $\text{supp} f \subset U$ , it holds that  $\langle \Lambda, f \rangle = 0$ .

We note that by a well-known Gaussian estimate for the transition density of  $X$  (the special case  $J \equiv 1$  and  $n = 0$  in Lemma 3.8), we have

$$\sup_{t \in K} \mathbf{E}[\exp(r|X(t, x, w)|)] < +\infty$$

for all  $r > 0$  and compact set  $K \subset (0, \infty)$ . Hence by Corollary 2.7, for any  $\Lambda = \partial_1^k \cdots \partial_d^k [\exp(k|x|)f(x)] \in \mathcal{E}'(\mathbb{R}^d)$ , the mapping  $(0, \infty) \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_p^{-kd}$  is continuous for every  $p \in (1, \infty)$ .

**Lemma 3.9.** *Let  $x \in \mathbb{R}^d$  and assume (H1) and (H2). Then for every  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$  with  $\text{supp} \Lambda \not\ni x$ , there exists  $k \in \mathbb{Z}_{\geq 0}$  such that*

$$\lim_{t \downarrow 0} \|\Lambda(X(t, x, w))\|_{p, -k} = 0 \quad \text{for each } p \in (1, \infty).$$

From this, the following is immediate.

**Proposition 3.10.** *Let  $x \in \mathbb{R}^d$  and assume (H1) and (H2). Then for every  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$  with  $\text{supp} \Lambda \not\ni x$ , there exists  $k \in \mathbb{Z}_{\geq 0}$  such that*

$$\int_0^T \|\Lambda(X(t, x, w))\|_{p, -k}^p dt < +\infty \quad \text{for each } p \in (1, \infty).$$

*Proof of Lemma 3.9.* We assume  $d = 1$  just for simplicity. Let  $p \in (1, \infty)$  be arbitrary. Assume  $\Lambda = \frac{d^k}{dx^k} [\exp(k|x|)f(x)]$ , where  $k$  is a nonnegative integer and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded continuous function.

Since  $x \notin \text{supp} \Lambda$ , we have  $\delta_0 := \inf\{|x - y| : y \in \text{supp} \Lambda\} > 0$ . Let  $\Omega := \cup_{y \in \text{supp} \Lambda} B_{\delta_0/2}(y)$  be the  $(\delta_0/2)$ -neighbourhood of  $\text{supp} \Lambda$ , where  $B_r(y)$  is the open ball with center  $y$  and radius  $r$ . Then, we have  $\Lambda \in \mathcal{D}'(\Omega)$ , and the function  $f$  can be rearranged so that  $\text{supp} f \subset \Omega$ .

Now let  $\varepsilon > 0$  and  $J \in \mathbb{D}^\infty$  be arbitrary. We have

$$\begin{aligned} \mathbf{E}[\Lambda(X^\varepsilon(1, x, w))J] &= \mathbf{E}[(e_k f)^{(k)}(X^\varepsilon(1, x, w))J] \\ &= \mathbf{E}[\exp(k|X^\varepsilon(1, x, w)|)f(X^\varepsilon(1, x, w))l_\varepsilon(J)] \end{aligned}$$

where  $l_\varepsilon(J) \in \mathbb{D}^\infty$  is of the form

$$l_\varepsilon(J) \in \mathbb{D}^\infty = \sum_{j=0}^k \langle P_j(\varepsilon, w), D^j J \rangle_{H^{\otimes j}}$$

for some  $P_j(\varepsilon, w) \in \mathbb{D}^\infty(H^{\otimes j})$ ,  $j = 0, 1, \dots, k$ , which are polynomials in  $F(\varepsilon, x, w)$ , its derivatives up to the order  $k$ , and  $|DX^\varepsilon(t, x, w)|_H^{-2}$ . Take  $q' \in (1, q)$ . Since  $\{F(\varepsilon, x, w)\}_{\varepsilon > 0}$  is uniformly non-degenerate, there exists  $c_0, \nu > 0$  such that

$$\|l_\varepsilon(J)\|_{q'} \leq c_0 \varepsilon^{-\nu} \|J\|_{q, k} \quad \text{for all } \varepsilon \in (0, T] \text{ and } J \in \mathbb{D}^\infty.$$

Therefore by taking  $p' \in (1, \infty)$  such that  $1/p' + 1/q' = 1$ , we have

$$\begin{aligned} & |\mathbf{E}[\Lambda(X^\varepsilon(1, x, w))J]| \\ & \leq \|\exp(k|X^\varepsilon(1, x, w)|)f(X^\varepsilon(1, x, w))\|_{p'} \|l_\varepsilon(J)\|_{q'} \\ & \leq c_0 \varepsilon^{-\nu} \|\exp(k|X^\varepsilon(1, x, w)|)f(X^\varepsilon(1, x, w))\|_{p'} \|J\|_{q,k}, \end{aligned}$$

which implies  $\|\Lambda(X^\varepsilon(1, x, w))\|_{p,-k} \leq c_0 \varepsilon^{-\nu} \|\exp(k|X^\varepsilon(1, x, w)|)f(X^\varepsilon(1, x, w))\|_{p'}$  for any  $\varepsilon > 0$ . By Proposition 3.5, we obtain

$$\begin{aligned} & \|\Lambda(X(t, x, w))\|_{p,-k} = \|\Lambda(X^{\sqrt{t}}(1, x, w))\|_{p,-k} \\ & \leq c_0 t^{-\nu/2} \|\exp(k|X^{\sqrt{t}}(1, x, w)|)f(X^{\sqrt{t}}(1, x, w))\|_{p'} \\ & = c_0 t^{-\nu/2} \|\exp(k|X_t|)f(X_t)1_{\{|X_t - x| > \frac{\delta_0}{2}\}}\|_{p'}. \end{aligned}$$

By Lemma 3.8 (the special case where  $n = 0$ ), we find that for any  $r > 0$ ,  $\mathbf{E}[\exp(r|X(t, x, w)|)] = O(t^{-\nu'})$  as  $t \downarrow 0$  for some  $\nu' > 0$  (actually this is  $O(1)$  because of the well-known Gaussian estimate for the transition density). However we have  $\mathbf{P}(|X_t - x| > \frac{\delta_0}{2}) = O(e^{-\delta_0^2/(c_1 t)})$  as  $t \downarrow 0$  for some constant  $c_1 > 0$ , so that the last quantity converges to zero as  $t \downarrow 0$ , and hence get the conclusion.  $\square$

**3.3. Hölder continuity of local time in space variable: a special case.** In this subsection, we assume  $d = 1$ , (H1) and (H3). Let  $X = (X_t)_{t \geq 0}$  be a unique strong solution to the following one-dimensional stochastic differential equation

$$(3.15) \quad dX_t = \sigma(X_t)dw(t) + \frac{1}{2}\sigma(X_t)\sigma'(X_t)dt, \quad X_0 = x \in \mathbb{R},$$

or equivalently,

$$dX_t = \sigma(X_t) \circ dw(t), \quad X_0 = x.$$

The main purpose in this subsection is to prove

**Theorem 3.11.** *Suppose that  $\sigma$  satisfies (H1) and (H3), and let  $X = (X_t)_{t \geq 0}$  be the unique strong solution to (3.15). Then for each  $s < \frac{1}{2}$  and  $\beta \in (0, \min\{\frac{1}{2} - s, 1\})$ , there exists a constant  $c = c(s, \beta) > 0$  such that*

$$\left\| \sigma(y) \int_0^1 \delta_y(X_t)dt - \sigma(z) \int_0^1 \delta_z(X_t)dt \right\|_{2,s} \leq c|y - z|^\beta$$

for every  $y, z \in \mathbb{R}$ .

Note that the object  $\sigma(y)^2 \int_0^t \delta_y(X_u)du$  is identified with the symmetric local time associated to the diffusion process  $(X_t)_{t \geq 0}$ . See Remark 4.3.

The Hermite polynomials  $H_n$ ,  $n \in \mathbb{Z}_{\geq 0}$  is defined by Define  $H_0(x) = 1$  and  $H_n(x) := \partial^{*n}1(x)$  for  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ , where the operator  $\partial^*$  is given by

$$\partial^* f(x) := -f'(x) + xf(x), \quad x \in \mathbb{R}$$

for any differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

We shall recall the following.

**Lemma 3.12.** *For each  $n \in \mathbb{Z}_{\geq 0}$ ,*

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \frac{(-1)^n e^{\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (i\xi)^n e^{-\frac{\xi^2}{2}} e^{i\xi x} d\xi.$$

The proof of Theorem 3.11 starts from this paragraph. Let  $y, z \in \mathbb{R}$  be arbitrary. Let  $A = A_z := \sigma(z) \frac{d}{dz}$  and  $p_t(z_1, z_2)$  be the transition-density function of  $X$ . For each  $t \geq 0$ , the Krylov-Veretennikov formula tells us that  $\delta_a(X_t) = \sum_{n=0}^{\infty} J_n[\delta_a(X_t)]$  (the convergence is in  $\mathbb{D}_2^{-\infty}$ ) for every  $a \in \mathbb{R}$ , where

(3.16)

$$J_n[\delta_a(X_t)] = \int_{0 \leq t_1 < \dots < t_n \leq t} \Pi_n(x; t, a)[t_1, \dots, t_n] dw(t_1) \cdots dw(t_n)$$

and

$$\begin{aligned} & \Pi_n(x; t, a)[t_1, \dots, t_n] \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} p_{t_1}(x, z_1) A_{z_1}[p_{t_2-t_1}(z_1, z_2)] \cdots A_{z_n}[p_{t-t_n}(z_n, a)] dz_1 \cdots dz_n. \end{aligned}$$

We remark that the stochastic integral in (3.16) is well defined since the square-integrability of each component  $(t_1, \dots, t_n) \mapsto \Pi_n(x; t, a)[t_1, \dots, t_n]$  of the chaos kernel  $\{\Pi_n(x; t, a)\}_{n=0}^{\infty}$  is established in [29, Corollary 4.5], in a more general situation. Since the generator  $L = \frac{1}{2}A^2$  commutes with  $A$ , the formula (3.16) reduces to

$$J_n[\delta_a(X_t)] = [A_x^n p_t(x, a)] \int_{0 \leq t_1 < \dots < t_n \leq t} dw(t_1) \cdots dw(t_n).$$

Therefore, by using Corollary 3.4, we have

$$\sigma(a) \int_0^1 \delta_a(X_s) ds = \sum_{n=0}^{\infty} \int_0^1 [\sigma(a) A_x^n p_t(x, a)] \int_{0 \leq t_1 < \dots < t_n \leq t} dw(t_1) \cdots dw(t_n) dt.$$

Hence we have

$$\begin{aligned} & \left\| \sigma(y) \int_0^1 \delta_y(X_t) dt - \sigma(z) \int_0^1 \delta_z(X_t) dt \right\|_{2,s}^2 \\ &= \sum_{n=0}^{\infty} (1+n)^s \mathbf{E} \left[ \left\{ \int_0^1 [\sigma(y) A_x^n p_t(x, y) - \sigma(z) A_x^n p_t(x, z)] \right. \right. \\ & \quad \left. \left. \times \int_{0 \leq t_1 < \dots < t_n \leq t} dw(t_1) \cdots dw(t_n) dt \right\}^2 \right], \end{aligned}$$

and so we need to compute

(3.17)

$$\begin{aligned}
I_n &:= \mathbf{E} \left[ \left\{ \int_0^1 [\sigma(y) A_x^n p_t(x, y) - \sigma(z) A_x^n p_t(x, z)] \int_{0 \leq t_1 < \dots < t_n \leq t} dw(t_1) \cdots dw(t_n) dt \right\}^2 \right] \\
&= 2 \int_{0 \leq s < t \leq 1} ds dt \left\{ \sigma(y) A_x^n p_s(x, y) - \sigma(z) A_x^n p_s(x, z) \right\} \left\{ \sigma(y) A_x^n p_t(x, y) - \sigma(z) A_x^n p_t(x, z) \right\} \\
&\quad \times \mathbf{E} \left[ \int_{0 \leq t_1 < \dots < t_n \leq s} dw(t_1) \cdots dw(t_n) \int_{0 \leq t_1 < \dots < t_n \leq t} dw(t_1) \cdots dw(t_n) \right] \\
&= \frac{2}{n!} \int_{0 \leq s < t \leq 1} s^n ds dt \left\{ \sigma(y) A_x^n p_s(x, y) - \sigma(z) A_x^n p_s(x, z) \right\} \left\{ \sigma(y) A_x^n p_t(x, y) - \sigma(z) A_x^n p_t(x, z) \right\}.
\end{aligned}$$

Since  $\sigma$  is Lipschitz (which is because of (H1)), the vector field  $A = \sigma(z)(d/dz)$  is *complete*, so that one can associate the one-parameter group of diffeomorphisms  $\{e^{sA}\}_{s \in \mathbb{R}}$ .

**Lemma 3.13.**  $\frac{d}{du} e^{uA}(x) = \sigma(x) \frac{\partial}{\partial x} e^{uA}(x)$  for every  $u \in \mathbb{R}$  and  $x \in \mathbb{R}$ .

*Proof.* By the homomorphism property:  $e^{(s+u)A} = e^{sA} \circ e^{uA}$ , we have  $\frac{d}{ds} e^{sA}(x) = \frac{d}{du} \big|_{u=0} e^{(s+u)A}(x) = \frac{d}{du} \big|_{u=0} e^{sA}(e^{uA}(x)) = \sigma(x) \frac{\partial}{\partial x} e^{sA}(x)$ .  $\square$

To continue the calculation of  $I_n$ , we shall take a look at  $A_x p_t(x, a)$ . The unique strong solution  $X = (X_t)_{t \geq 0}$  is now expressed by  $X_t = e^{w(t)A}(x)$ . Therefore by using Lemma 3.13, we have

$$\begin{aligned}
A_x p_t(x, a) &= A_x \mathbf{E}[\delta_a(e^{w(t)A}(x))] \\
&= \sigma(x) \frac{\partial}{\partial x} \int_{\mathbb{R}} \delta_a(e^{\sqrt{t}uA}(x)) \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \\
&= \int_{\mathbb{R}} \left( \sigma(x) \frac{\partial}{\partial x} e^{\sqrt{t}uA}(x) \right) \delta'_a(e^{uA}(x)) \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \\
&= \int_{\mathbb{R}} \left( \frac{1}{\sqrt{t}} \frac{d}{du} e^{\sqrt{t}uA}(x) \right) \delta'_a(e^{\sqrt{t}uA}(x)) \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \\
&= \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \left( \frac{d}{du} \delta_a(e^{\sqrt{t}uA}(x)) \right) \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \\
&= \int_{\mathbb{R}} \delta_a(e^{\sqrt{t}uA}(x)) \left( \frac{-1}{\sqrt{t}} \frac{d}{du} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \right) du,
\end{aligned}$$

where the integral is understood as the coupling of the Schwartz distribution and the test function. By the repetition of the above procedure, we obtain

$$\begin{aligned}
A_x^n p_t(x, a) &= \int_{\mathbb{R}} \delta_a(e^{\sqrt{t}uA}(x)) \left\{ \frac{(-1)^n}{t^{n/2}} \frac{d^n}{du^n} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \right\} du \\
&= \int_{\mathbb{R}} \delta_a(e^{\sqrt{t}uA}(x)) \frac{1}{t^{n/2}} H_n(u) \frac{e^{-u^2/2}}{\sqrt{2\pi}} du.
\end{aligned}$$

For each  $t > 0$ , define a mapping  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  by  $\varphi(u) := e^{\sqrt{t}uA}(x)$ . Since  $\varphi$  is continuously differentiable and  $|\varphi'(u)| = \sqrt{t}|\sigma(\varphi(u))| \geq \sqrt{t}\lambda^{1/2}$  for every  $u$ , where  $\lambda > 0$  is what appeared in (H3), we see that  $\varphi$  is a diffeomorphism. Hence we can apply the change of variables  $w = \varphi(u)$ <sup>†</sup> (and then  $du/dw = (\sqrt{t}\sigma(w))^{-1}$ ) in the above integral to get

$$\begin{aligned} A_x^n p_t(x, a) &= \int_{\mathbb{R}} \delta_a(w) \frac{1}{t^{(n+1)/2}} H_n(\varphi^{-1}(w)) \frac{e^{-(\varphi^{-1}(w))^2/2}}{\sqrt{2\pi}\sigma(w)} dw \\ &= \frac{1}{t^{(n+1)/2}} H_n(\varphi^{-1}(a)) \frac{e^{-(\varphi^{-1}(a))^2/2}}{\sqrt{2\pi}\sigma(a)}. \end{aligned}$$

By using Lemma 3.12, we obtain the formula

$$(3.18) \quad A_x^n p_t(x, a) = \frac{(-1)^n t^{-(n+1)/2}}{2\pi\sigma(a)} \int_{-\infty}^{\infty} (i\xi)^n e^{-\xi^2/2} e^{i\xi\varphi^{-1}(a)} d\xi.$$

where  $\varphi(u) = e^{\sqrt{t}uA}(x)$ .

Therefore by using (3.18),

$$\begin{aligned} (3.19) \quad & \sigma(y) A_x^n p_t(x, y) - \sigma(z) A_x^n p_t(x, z) \\ &= \frac{(-1)^n t^{-(n+1)/2}}{2\pi} \int_{-\infty}^{\infty} (i\xi)^n e^{-\xi^2/2} \{e^{i\xi\varphi^{-1}(y)} - e^{i\xi\varphi^{-1}(z)}\} d\xi \\ &=: \mathbb{I}_n. \end{aligned}$$

**Lemma 3.14.** *For any  $\alpha \in [0, 1]$  and  $\theta \in \mathbb{R}$ , we have  $|e^{i\theta} - 1| \leq 2|\theta|^\alpha$  with the convention  $0^0 := 1$ .*

*Proof.* Let  $\alpha \in [0, 1]$ . If  $\theta \in \mathbb{R} \setminus (-1, 1)$ , then we see that  $|e^{i\theta} - 1| \leq 2 \leq 2|\theta|^\alpha$ . On the other hand, for  $\theta \in (-1, 1)$ ,  $|e^{i\theta} - 1| \leq |\theta| \leq |\theta|^\alpha \leq 2|\theta|^\alpha$  because  $\alpha \in [0, 1]$ . Thus we obtain  $|e^{i\theta} - 1| \leq 2|\theta|^\alpha$  for all  $\theta \in \mathbb{R}$ .  $\square$

Let  $\beta \in [0, 1)$  be arbitrary. By Lemma 3.14, we have

$$\begin{aligned} |\mathbb{I}_n| &\leq \frac{t^{-(n+1)/2}}{\pi} |\varphi^{-1}(y) - \varphi^{-1}(z)|^\beta \int_{-\infty}^{\infty} \xi^{n+\beta} e^{-\xi^2/2} d\xi \\ &= \frac{t^{-(n+1)/2} 2^{(n+\beta-1)/2}}{\pi} \Gamma\left(\frac{n+\beta+1}{2}\right) |\varphi^{-1}(y) - \varphi^{-1}(z)|^\beta. \end{aligned}$$

Since  $\frac{d}{dw} \varphi^{-1}(w) = \frac{1}{\sqrt{t}\sigma(w)}$ , we have  $|\varphi^{-1}(y) - \varphi^{-1}(z)| = t^{-1/2} \left| \int_y^z \sigma(w)^{-1} dw \right|$ , so that

$$(3.20) \quad |\mathbb{I}_n| \leq \frac{t^{-(n+\beta+1)/2} 2^{(n+\beta-1)/2}}{\pi \lambda^{\beta/2}} \Gamma\left(\frac{n+\beta+1}{2}\right) |y - z|^\beta,$$

where it is recalled again that  $\lambda > 0$  is what appeared in (H3).

---

<sup>†</sup>when  $t = 1$ , we see that  $\varphi^{-1}(z) = \int_x^z \frac{da}{\sigma(a)}$  and the stochastic process  $\varphi^{-1}(X_s)$ , which is nothing but the Wiener process  $w(s)$ , is known as the Lamperti transformation of  $X$ .



Substituting (3.20) into (3.19),

$$\begin{aligned}
& |\sigma(y)A_x^n p_t(x, y) - \sigma(z)A_x^n p_t(x, z)| \\
& \leq \frac{t^{-(n+\beta+1)/2} 2^{(n+\beta-1)/2}}{\pi \lambda^{\beta/2}} \Gamma\left(\frac{n+\beta+1}{2}\right) |y-z|^\beta \\
& = c_2 t^{-(n+\beta+1)/2} 2^{(n+\beta-1)/2} \Gamma\left(\frac{n+\beta+1}{2}\right) |y-z|^\beta
\end{aligned}$$

where  $c_2 := (\pi \lambda^{\beta/2})^{-1}$ . Note that  $c_2$  does not depend on  $y$  and  $z$ . With putting

$$c_3(n) := \Gamma\left(\frac{n+\beta+1}{2}\right),$$

we have obtained

$$\begin{aligned}
(3.21) \quad & |\sigma(y)A_x^n p_t(x, y) - \sigma(z)A_x^n p_t(x, z)| \leq c_2 t^{-(n+\beta+1)/2} 2^{(n+\beta-1)/2} c_3(n) |y-z|^\beta.
\end{aligned}$$

Now, by substituting (3.21) into (3.17), we have

$$\begin{aligned}
I_n &= \frac{2}{n!} \int_{0 \leq s < t \leq 1} s^n ds dt \\
&\quad \times \{ \sigma(y)A_x^n p_s(x, y) - \sigma(z)A_x^n p_s(x, z) \} \{ \sigma(y)A_x^n p_t(x, y) - \sigma(z)A_x^n p_t(x, z) \} \\
&\leq \frac{(c_2)^2}{n!} 2^{n+\beta} c_3(n)^2 |y-z|^{2\beta} \int_0^1 s^n s^{-(n+\beta+1)/2} ds \int_s^1 t^{-(n+\beta+1)/2} dt.
\end{aligned}$$

Note that the last iterated integral is finite because  $\beta < 1$ , and gives

$$\int_0^1 s^n s^{-(n+\beta+1)/2} ds \int_s^1 t^{-(n+\beta+1)/2} dt = \frac{2}{(1-\beta)(n-\beta+1)}.$$

Hence we have

$$I_n \leq \frac{(c_2)^2}{(1-\beta)} |y-z|^{2\beta} \frac{2^{n+\beta+1} c_3(n)^2}{n!(n-\beta+1)}.$$

Finally we have

$$\begin{aligned}
& \left\| \sigma(y) \int_0^1 \delta_y(X_t) dt - \sigma(z) \int_0^1 \delta_z(X_t) dt \right\|_{2,s}^2 = \sum_{n=0}^{\infty} (1+n)^2 I_n \\
& \leq \frac{(c_2)^2}{(1-\beta)} |y-z|^{2\beta} \sum_{n=0}^{\infty} (1+n)^s \frac{2^{n+\beta+1} c_3(n)^2}{n!(n-\beta+1)}.
\end{aligned}$$

By Stirling's formula, we see that the quantity

$$(1+n)^s \frac{2^{n+\beta+1} c_3(n)^2}{n!(n-\beta+1)} = (1+n)^s \frac{2^{n+\beta+1}}{n!(n-\beta+1)} \Gamma\left(\frac{n+\beta+1}{2}\right)^2$$

behaves like

$$\begin{aligned} & (1+n)^s \frac{2^{n+\beta+1}}{(n-\beta+1)\sqrt{2\pi n}(\frac{n}{e})^n} \times \left\{ \sqrt{\pi(n+\beta-1)} \left( \frac{n+\beta-1}{2e} \right)^{\frac{n+\beta-1}{2}} \right\}^2 \\ &= \frac{(1+n)^s 2^{n+\beta+1}}{(n-\beta+1)\sqrt{2\pi n}(\frac{n}{e})^n} \pi(n+\beta-1) \left( \frac{n+\beta-1}{2e} \right)^{n+\beta-1} = O(n^{s+\beta-\frac{3}{2}}) \end{aligned}$$

as  $n \rightarrow \infty$  for each  $\beta$ . Hence the sum converges if  $s + \beta - \frac{3}{2} < -1$ , i.e.,  $s + \beta < \frac{1}{2}$ . The proof of Theorem 3.11 finishes.

**Corollary 3.15.** *Assume (H1) and (H3). Let  $p_t(x, y)$  be the transition density of  $X_t$ . Then the function*

$$\mathbb{R} \ni y \mapsto \sigma(y) \int_0^1 p_t(x, y) dt \in \mathbb{R}$$

*is (globally)  $\beta$ -Hölder continuous for every  $\beta < 1$ .*

*Proof.* This is clear from Theorem 3.11 and the inequality

$$\begin{aligned} & \left| \sigma(y) \int_0^1 p_t(x, y) dt - \sigma(z) \int_0^1 p_t(x, z) dt \right| \\ &= \left| \mathbf{E}[\sigma(y) \int_0^1 \delta_y(X_t) dt - \sigma(z) \int_0^1 \delta_z(X_t) dt] \right| \\ &\leq \left\| \sigma(y) \int_0^1 \delta_y(X_t) dt - \sigma(z) \int_0^1 \delta_z(X_t) dt \right\|_{2, -1/2}. \end{aligned}$$

□

#### 4. ITÔ'S FORMULA FOR GENERALIZED WIENER FUNCTIONALS

Let  $w = (w(t))_{t \geq 0}$  be the  $d$ -dimensional Wiener process with  $w(0) = 0$  and let  $(\mathcal{F}_t^w)_{t \geq 0}$  be the filtration generated by  $w$ :  $\mathcal{F}_t^w := \sigma(w(s) : 0 \leq s \leq t)$ , for  $t \geq 0$ . Similarly to the previous section, we fix  $x \in \mathbb{R}^d$  and consider the following  $d$ -dimensional stochastic differential equation

$$(4.1) \quad dX_t = \sigma(X_t)dw(t) + b(X_t)dt, \quad X_0 = x \in \mathbb{R}^d.$$

Also in this section, we assume the conditions (H1) and (H2). We denote by  $\{X(t, x, w)\}_{t \geq 0}$  a unique strong solution  $X = (X_t)_{t \geq 0}$  to (4.1).

As mentioned before (in front of Lemma 3.9), we have

$$\sup_{t \in K} \mathbf{E}[\exp(r|X(t, x, w)|)] < +\infty$$

for all  $r > 0$  and compact set  $K \subset (0, \infty)$  (which one can deduce from the special case  $n = 0$  in Lemma 3.8). Hence by Corollary 2.7, for any  $p \in (1, \infty)$  and  $\Lambda = \partial_1^k \cdots \partial_d^k [\exp(k|x|)f(x)] \in \mathcal{E}'(\mathbb{R}^d)$ , the mapping  $(0, \infty) \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_p^{-k}$  is continuous.

**4.1. Stochastic integrals of pull-backs by diffusion.** Throughout this subsection, we fix  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$ . For each  $J \in \mathbb{D}^\infty$ , we will denote by  $(DJ)^i$  the  $i$ -th component of  $DJ \in \mathbb{D}^\infty(H)$ :  $DJ = ((DJ)^1, \dots, (DJ)^d)$ . For each  $t \in [0, T]$ , the evaluation map  $\text{ev}_t : H \ni h \mapsto h(t) \in \mathbb{R}^d$  naturally induces a map  $\text{id} \otimes \text{ev}_t : L_2(H) \cong L_2 \otimes H \rightarrow L_2 \otimes \mathbb{R}^d$  and then we write  $D_t J := ((D_t J)^1, \dots, (D_t J)^d) := \frac{d}{dt}(\text{id} \otimes \text{ev}_t)(DJ)$  for a.a.  $t \in [0, T]$ .

Let  $k \in \mathbb{N}$  and suppose that  $\int_0^T \|\Lambda(X(t, x, w))\|_{2, -k}^2 dt < +\infty$ . We define the *stochastic integrals*  $\int_0^T \Lambda(X(t, x, w)) dw^i(t)$ ,  $i = 1, \dots, d$  as elements in  $\mathbb{D}^{-\infty}$  via the pairing

(4.2)

$$\mathbf{E}\left[\left(\int_0^T \Lambda(X(t, x, w)) dw^i(t)\right) J\right] = \int_0^T \mathbf{E}[\Lambda(X(t, x, w))(D_t J)^i] dt,$$

for  $J \in \mathbb{D}^\infty$  and  $i = 1, \dots, d$ . We define the stochastic integral  $\int_s^T \Lambda(X(t, x, w)) dw^i(t)$  similarly for each  $0 < s \leq T$ .

This pairing is well defined because of the following:

**Proposition 4.1.** *For each  $k \in \mathbb{N}$  and  $i = 1, \dots, d$ , there exists a constant  $C > 0$  such that*

$$\int_0^T |\mathbf{E}[\Lambda(X(t, x, w))(D_t J)^i]| dt \leq C \left\{ \int_0^T \|\Lambda(X(t, x, w))\|_{2, -k}^2 dt \right\}^{1/2} \|J\|_{2, k+1}$$

for all  $J \in \mathbb{D}^\infty$ .

*Proof.* For simplicity of notation, we prove in the case  $d = 1$ . For each  $k \in \mathbb{N}$  and  $J \in \mathbb{D}^\infty$ ,

$$\begin{aligned} (4.3) \quad & \int_0^T |\mathbf{E}[\Lambda(X_t) D_t J]| dt \leq \int_0^T \|\Lambda(X_t)\|_{2, -k} \|D_t J\|_{2, k} dt \\ & \leq \left\{ \int_0^T \|\Lambda(X_t)\|_{2, -k}^2 dt \right\}^{1/2} \left\{ \int_0^T \|D_t J\|_{2, k}^2 dt \right\}^{1/2}. \end{aligned}$$

By Meyer's inequality, there exist constants  $c', C' > 0$  such that

$$c' \|D^{k'} F\|_2 \leq \|F\|_{2, k'} \leq C' \sum_{l=0}^{k'} \|D^l F\|_2, \quad \text{for all } F \in \mathbb{D}_2^{k'}$$

for  $k' = 1, 2, \dots, k+1$ . Therefore we have

$$\begin{aligned} \|D_t J\|_{2, k}^2 & \leq (C')^2 \left\{ \sum_{l=0}^k \|D^l D_t J\|_2 \right\}^2 \leq (C')^2 (k+1) \sum_{l=0}^k \|D^l D_t J\|_2^2 \\ & = (C')^2 (k+1) \sum_{l=0}^k \mathbf{E}[\|D^l D_t J\|_{H^{\otimes l}}^2] \\ & = C'' \left\{ \mathbf{E}[(D_t J)^2] + \sum_{l=1}^k \int_0^T \cdots \int_0^T \mathbf{E}[(D_{s_l} \cdots D_{s_1} D_t J)^2] ds_1 \cdots ds_l \right\}, \end{aligned}$$

where  $C'' := (C')^2(k+1)$ , so that

$$\begin{aligned} & \int_0^T \|D_t J\|_{2,k}^2 dt \\ & \leq C'' \sum_{l=1}^{k+1} \int_0^T \cdots \int_0^T \mathbf{E}[(D_{u_l} \cdots D_{u_1} J)^2] du_1 \cdots du_l \\ & \leq C'' \sum_{l=0}^{k+1} \|D^l J\|_2^2 \leq (c')^{-2} C'' \|J\|_{2,k+1}^2. \end{aligned}$$

Hence by substituting this into (4.3), we get the result.  $\square$

**Remark 4.1.** (a) As is easily seen, the stochastic integral  $\int_0^T \Lambda(X_t) dw^i(t)$  has another expression:

$$\int_0^T \Lambda(X_t) dw^i(t) = D^*[(0, \dots, \underbrace{\int_0^\bullet \Lambda(X_u) du}_{i\text{-th position}}, \dots, 0)].$$

Hence it coincides with the Skorokhod integral as long as the object inside the  $D^*$  lies in the domain  $\mathbb{D}_2^1(H)$  and then automatically coincides with the Itô integral because of the adaptedness. In fact, in view of the Clark-Ocone formula, every  $J \in L_2$  can be written as  $J = \mathbf{E}[J] + \sum_{i=1}^d \int_0^T \mathbf{E}[(D_t J)^i | \mathcal{F}_t^w] dw^i(t)$  and then (4.2) is just the Itô isometry. Therefore, our stochastic integral might be seemingly a natural extension of the classical anticipative stochastic integral to this distributional setting.

(b) By Proposition 4.1, we have  $\int_0^T \Lambda(X_t) dw^i(t) \in \mathbb{D}_2^{-(k+1)}$  for  $i = 1, \dots, d$  provided  $\int_0^T \|\Lambda(X_t)\|_{2,-k}^2 dt < +\infty$ .

The following is the main result in this subsection. This is a version of what was due to Uemura [23, Proposition 1].

**Theorem 4.2** (cf. Uemura [23]). *Let  $s \in \mathbb{R}$ ,  $p \geq 2$  and assume that  $(0, T] \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_p^s$  is continuous. Then we have*

$$\int_0^T \Lambda(X(t, x, w)) dw^i(t) \in \mathbb{D}_p^s, \quad \text{for } i = 1, \dots, d$$

*provided either one of the following*

- (i)  $\lim_{t \downarrow 0} \|\Lambda(X(t, x, w))\|_{p,s} = 0$ .
- (ii)  $s \geq 0$  and  $\int_0^T \|\Lambda(X(t, x, w))\|_{p,s}^2 dt < \infty$ .

**Remark 4.2.** (a) See also a remark just after Lemma 4.10 for verification of the continuity assumption.

(b) From the proof of Theorem 4.2, we would find that

$$\int_{t_0}^T \Lambda(X(t, x, w)) dw^i(t) \in \mathbb{D}_p^s \quad \text{for any } t_0 > 0$$

and  $i = 1, \dots, d$  if  $(0, T] \ni t \mapsto \Lambda(X_t) \in \mathbb{D}_p^s$  is continuous.

The proof of Theorem 4.2 mainly consists of the following series of Propositions 4.4, 4.5 and 4.6. We will give the proof at the last of this subsection.

Before the next definition, we note that  $\mathbf{E}[F|\mathcal{F}_t^w] \in \mathbb{D}^\infty$  for every  $t \geq 0$  if  $F \in \mathbb{D}^\infty$ .

**Definition 4.3.** Let  $t \geq 0$ . We say that a generalized Wiener functional  $F \in \mathbb{D}^{-\infty}$  is  $\mathcal{F}_t^w$ -measurable if it holds that  $\mathbf{E}[FG] = \mathbf{E}[F\mathbf{E}[G|\mathcal{F}_t^w]]$  for any  $G \in \mathbb{D}^\infty$ .

**Proposition 4.4.** Let  $s \in \mathbb{R}$  and  $p \geq 2$ . Then there exists  $c = c(p, s) > 0$  such that, for any

- (i) mapping  $F : (0, T] \ni t \mapsto F(t) \in \mathbb{D}_p^s$  with  $F(t)$  is  $\mathcal{F}_t^w$ -measurable for any  $t \in (0, T]$ ,
- (ii) division  $0 = t_0 < t_1 < \dots < t_n = T$ , and
- (iii)  $i = 1, \dots, d$ ,

we have

$$\left\| \sum_{k=2}^n F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})) \right\|_{p,s}^p \leq c \sum_{k=2}^n \|F(t_{k-1})\|_{p,s}^p (t_k - t_{k-1}).$$

*Proof.* Let  $0 = t_0 < t_1 < \dots < t_n = T$  be any division of  $[0, T]$  and set

$$\Phi := \sum_{k=2}^n F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})).$$

To calculate  $\|\Phi\|_{p,s}$ , we begin with the chaos expansion of each  $F(t_{k-1})(w^i(t_k) - w^i(t_{k-1}))$ . Noting  $\mathbf{E}[F(t_{k-1})(w^i(t_k) - w^i(t_{k-1}))] = 0$ , one finds that the chaos expansion is given by

$$F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})) = \sum_{m=1}^{\infty} J_{m-1}[F(t_{k-1})](w^i(t_k) - w^i(t_{k-1})),$$

where  $F(t_{k-1}) = \sum_{m=0}^{\infty} J_m[F(t_{k-1})]$  is the chaos expansion of  $F(t_{k-1})$ . Hence we have

$$\begin{aligned} & (I - \mathcal{L})^{s/2} \sum_{k=2}^n F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})) \\ &= \sum_{k=2}^n \sum_{m=1}^{\infty} (1+m)^{s/2} J_{m-1}[F(t_{k-1})](w^i(t_k) - w^i(t_{k-1})) \\ &= \sum_{m=0}^{\infty} \frac{(2+m)^{s/2}}{(1+m)^{s/2}} \sum_{k=2}^n (1+m)^{s/2} J_m[F(t_{k-1})](w^i(t_k) - w^i(t_{k-1})). \end{aligned}$$

By Meyer's  $L_p$ -multiplier theorem (see e.g. [11, Chapter V, Section 8, Lemma 8.2]), there exists  $c' = c'(p, s) > 0$  such that

$$\begin{aligned}
& \left\| \sum_{k=2}^n F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})) \right\|_{p,s} \\
&= \left\| (I - \mathcal{L})^{s/2} \sum_{k=2}^n F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})) \right\|_p \\
&\leq c' \left\| \sum_{m=0}^{\infty} \sum_{k=2}^n (1+m)^{s/2} J_m[F(t_{k-1})](w^i(t_k) - w^i(t_{k-1})) \right\|_p \\
&= c' \left\| \sum_{k=2}^n [(I - \mathcal{L})^{s/2} F(t_{k-1})](w^i(t_k) - w^i(t_{k-1})) \right\|_p.
\end{aligned}$$

Note that  $(I - \mathcal{L})^{s/2} F(t_{k-1}) \in L_2$  and is  $\mathcal{F}_{t_{k-1}}^w$ -measurable. Hence the last quantity in the  $L_p$ -norm can be written as

$$\int_0^T \sum_{k=2}^n [(I - \mathcal{L})^{s/2} F(t_{k-1})] 1_{(t_{k-1}, t_k]}(t) dw^i(t).$$

Thus by using the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
& \left\| \sum_{k=2}^n F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})) \right\|_{p,s}^p \\
&\leq c'' \mathbf{E} \left[ \left\{ \int_0^T \sum_{k=2}^n [(I - \mathcal{L})^{s/2} F(t_{k-1})]^2 1_{(t_{k-1}, t_k]}(t) dt \right\}^{p/2} \right]
\end{aligned}$$

for some constant  $c'' > 0$ . Finally, using the assumption  $p \geq 2$  and the Jensen inequality, we reached

$$\left\| \sum_{k=2}^n F(t_{k-1})(w^i(t_k) - w^i(t_{k-1})) \right\|_{p,s}^p \leq c \sum_{k=2}^n \|F(t_{k-1})\|_{p,s}^p (t_k - t_{k-1})$$

for some constant  $c > 0$ . □

Given  $n \in \mathbb{N}$ ,  $i = 1, \dots, d$  and the dyadic division  $\{t_k := kT/2^n\}_{k=0}^{2^n}$  of  $[0, T]$ , we define

$$\Phi_n := \sum_{k=2}^{2^n} \Lambda(X(t_{k-1}, x, w))(w^i(t_k) - w^i(t_{k-1})).$$

**Proposition 4.5.** *Let  $s \in \mathbb{R}$ ,  $p \geq 2$  and suppose that*

- (i)  $(0, T] \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_p^s$  is continuous and
- (ii)  $\lim_{t \downarrow 0} \|\Lambda(X(t, x, w))\|_{p,s} = 0$ .

*Then we have  $\|\Phi_n - \Phi_m\|_{p,s} \rightarrow 0$  as  $n, m \rightarrow \infty$ .*

*Proof.* Suppose that  $n < m$  and let  $t_k := kT/2^n$  and  $u_l := lT/2^m$ . Then we have

$$\Phi_n - \Phi_m = \sum_{k=2}^{2^n} \sum_{\substack{l \in \{0,1,\dots,2^m\}: \\ t_{k-1} < u_l \leq t_k}} [\Lambda(X_{t_{k-1}}) - \Lambda(X_{u_l})](w^i(u_l) - w^i(u_{l-1}))$$

and hence by Proposition 4.4, we obtain

$$\|\Phi_n - \Phi_m\|_{p,s}^p \leq \sum_{k=2}^{2^n} \sum_{\substack{l \in \{0,1,\dots,2^m\}: \\ t_{k-1} < u_l \leq t_k}} \|\Lambda(X_{t_{k-1}}) - \Lambda(X_{u_l})\|_{p,s}^p (u_l - u_{l-1}).$$

By the assumption, the mapping  $(0, T] \ni t \mapsto \Lambda(X_t) \in \mathbb{D}_p^s$  is uniformly continuous, from which, we easily get  $\|\Phi_n - \Phi_m\|_{p,s} \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**Proposition 4.6.** *Suppose  $k \in \mathbb{Z}_{\geq 0}$  and*

- (i)  $(0, T] \ni t \mapsto \Lambda(X(t, x, w)) \in \mathbb{D}_2^{-k}$  *is continuous;*
- (ii)  $\lim_{t \downarrow 0} \|\Lambda(X(t, x, w))\|_{2,-k} = 0$ .

*Then we have*

$$\|\Phi_n - \int_0^T \Lambda(X(t, x, w)) dw^i(t)\|_{2, -(k+1)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Proof.* The same argument in Proposition 4.1 leads us to

$$\begin{aligned} & \|\Phi_n - \int_0^T \Lambda(X_t) dw^i(t)\|_{2, -(k+1)} \\ & \leq \left\{ \int_0^{t_1} \|\Lambda(X_t)\|_{2,-k}^2 dt + \sum_{k=2}^{2^n} \int_{t_{k-1}}^{t_k} \|\Lambda(X_{t_{k-1}}) - \Lambda(X_t)\|_{2,-k}^2 dt \right\}^{1/2}. \end{aligned}$$

By the assumption, we have  $(0, T] \ni t \mapsto \Lambda(X_t) \in \mathbb{D}_2^{-k}$  is uniformly continuous, and hence the above quantity converges to zero as  $n \rightarrow \infty$ .  $\square$

*Proof of Theorem 4.2.* (i) Suppose that  $\lim_{t \downarrow 0} \|\Lambda(X_t)\|_{p,s} = 0$ . Firstly, suppose that  $s < 0$ . Let  $k$  be the greatest integer, not exceeding  $s$ , i.e.,  $k = \max\{k' \in \mathbb{Z} : k' \leq s\}$ . By the assumption, the stochastic integral  $\int_0^T \Lambda(X_t) dw^i(t)$  is defined as an element in  $\mathbb{D}_2^{k-1}$ . On the other hand, Proposition 4.5 tells us that  $\{\Phi_n\}_{n=1}^\infty$  is a Cauchy sequence in  $\mathbb{D}_p^s$  and hence converges to some  $\Phi \in \mathbb{D}_p^s$ . Hence  $\Phi_n$  converges to  $\Phi$  also in  $\mathbb{D}_2^{k-1}$ . Now by Proposition 4.6, it must be  $\int_0^T \Lambda(X_t) dw^i(t) = \lim_{n \rightarrow \infty} \Phi_n = \Phi \in \mathbb{D}_p^s$ .

Secondly, suppose that  $s \geq 0$ . In this case, it is clearly satisfied that  $\int_0^T \|\Lambda(X_t)\|_{p,s}^p dt < +\infty$ , and hence it reduces to the case (ii).



(ii) Suppose that  $s \geq 0$  and  $\int_0^T \|\Lambda(X_t)\|_{p,s}^p dt < +\infty$ . Then by Proposition 4.4, we have

$$\|\Phi_n\|_{p,s}^p \leq \sum_{k=2}^{2^n} \|\Lambda(X_{t_{k-1}})\|_{p,s}^p (t_k - t_{k-1}) \rightarrow \int_0^T \|\Lambda(X_t)\|_{p,s}^p dt,$$

as  $n \rightarrow \infty$ . Here,  $t_k = kT/2^n$ ,  $k = 0, 1, \dots, 2^n$ . Hence  $\{\Phi_n\}_{n=1}^\infty$  forms a bounded family in  $\mathbb{D}_p^s$ , so that by Alaoglu's theorem, there exists a subsequence  $\{\Phi_{n_l}\}_{l=1}^\infty$  and  $\Phi \in \mathbb{D}_p^s$  such that  $\Phi_{n_l} \rightarrow \Phi$  weakly in  $\mathbb{D}_p^s$ . In particular, since  $s \geq 0$  and  $p \geq 2$ , this convergence is still valid in the weak topology on  $L_2$ . On the other hand, it is clearly satisfied that  $\int_0^T \|\Lambda(X_t)\|_2^2 dt < \infty$ , and hence  $\{\Lambda(X_t)\}_{t \geq 0}$  is now a square-integrable  $(\mathcal{F}_t)_{t \geq 0}$ -adapted process. The classical stochastic analysis proves that  $\Phi_n \rightarrow \int_0^T \Lambda(X_t) dw^i(t)$  in  $L_2$ . Therefore, it must be  $\int_0^T \Lambda(X_t) dw^i(t) = \lim_{l \rightarrow \infty} \Phi_{n_l} = \Phi \in \mathbb{D}_p^s$ .  $\square$

**4.2. Distributional Itô's formula.** Let  $A_i$ ,  $i = 1, \dots, d$  and  $L$  be the vector fields and the second-order differential operator given by

$$\begin{aligned} (A_i f)(z) &:= \sum_{k=1}^d \sigma_i^k(z) \frac{\partial f}{\partial z_k}(z) \\ (L f)(z) &:= \frac{1}{2} \sum_{i,j=1}^d (\sigma^* \sigma)_i^j(z) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{i=1}^d b^i(z) \frac{\partial f}{\partial z_i}(z) \end{aligned}$$

for  $f \in \mathcal{S}(\mathbb{R}^d)$  and  $z \in \mathbb{R}^d$ . In the case of  $d = 1$ , the vector field  $A_1$  will be denoted by  $A$ . Under the assumption (H1), these operators naturally act on  $\mathcal{S}'(\mathbb{R}^d)$  and  $\mathcal{E}'(\mathbb{R}^d)$ .

**Theorem 4.7** (cf. Kubo [13]). *Let  $x \in \mathbb{R}^d$  and assume (H1) and (H2). Then for each  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$  and  $t_0 \in (0, T]$ , we have*

$$\begin{aligned} (4.4) \quad & \Lambda(X(T, x, w)) - \Lambda(X(t_0, x, w)) \\ &= \sum_{i=1}^d \int_{t_0}^T (A_i \Lambda)(X(t, x, w)) dw^i(t) + \int_{t_0}^T (L \Lambda)(X(t, x, w)) dt \quad \text{in } \mathbb{D}^{-\infty}. \end{aligned}$$

Similarly, we have (4.4) for  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$ .

*Proof.* For simplicity of notation, we assume  $d = 1$ . The case  $d \geq 2$  is similar. Fix  $t_0 > 0$ . Suppose that  $\Lambda \in \mathcal{S}_{-2k}$ . Then there exist  $\phi_n \in \mathcal{S}(\mathbb{R})$ ,  $n \in \mathbb{N}$  such that  $\Lambda = \lim_{n \rightarrow \infty} \phi_n$  in  $\mathcal{S}_{-2(k+1)}$ . By Itô's formula, we clearly have

$$\phi_n(X_T) - \phi_n(X_{t_0}) = \int_{t_0}^T (A \phi_n)(X_t) dw(t) + \int_{t_0}^T (L \phi_n)(X_t) dt$$

for each  $n \in \mathbb{N}$ . What we have to prove is the following: As  $n \rightarrow \infty$ ,

- (a)  $\|\Lambda(X_t) - \phi_n(X_t)\|_{2,-k} \rightarrow 0$  for  $t = t_0$  and  $T$ ,

$$\begin{aligned}
\text{(b)} \quad & \left\| \int_{t_0}^T \{A\Lambda(X_t) - A\phi_n(X_t)\} dw(t) \right\|_{2, -(k+2)} \rightarrow 0, \\
\text{(c)} \quad & \left\| \int_{t_0}^T \{L\Lambda(X_t) - L\phi_n(X_t)\} dt \right\|_{2, -(k+2)} \rightarrow 0.
\end{aligned}$$

It is easy to show (a). We shall prove (b). By Proposition 4.1, we have

$$\left\| \int_{t_0}^T \{A\Lambda(X_t) - A\phi_n(X_t)\} dw(t) \right\|_{2, -(k+2)}^2 \leq C \int_{t_0}^T \| (A(\Lambda - \phi_n))(X_t) \|_{2, -(k+1)}^2 dt.$$

Next we shall show that there exists  $c, \nu > 0$  such that

$$(4.5) \quad \| (A(\Lambda - \phi_n))(X_t) \|_{2, -(k+1)}^2 \leq ct^{-\nu} \| (\Lambda - \phi_n)(X_t) \|_{4, -k}^2$$

for every  $t \in [t_0, T]$ , and then the above quantities converge to zero uniformly in  $t \in [t_0, T]$  as  $n \rightarrow \infty$ , and hence (ii) is proved. To prove (4.5), it suffices to show that: there exist  $c, \nu > 0$  such that

$$(4.6) \quad |\mathbf{E}[(\sigma\Psi')(X_t)J]| \leq ct^{-\nu} \|\Psi(X_t)\|_{4, -k} \|J\|_{2, k+1}$$

for each  $\Psi \in \mathcal{S}_{-2k}$  and  $J \in \mathbb{D}^\infty$ . In fact, we have

$$\mathbf{E}[(\sigma\Psi')(X_t)J] = \mathbf{E}[\Psi(X_t) \left\{ P_0(t)\sigma(X_t)J + \langle P_1(t), D(\sigma(X_t)J) \rangle_H \right\}],$$

for some  $P_i(t) \in \mathbb{D}^\infty(H^{\otimes i})$ ,  $i = 0, 1$  which are polynomials in  $X_t = X(t, x, w)$ , its derivatives and  $\|DX(t, x, w)\|_H^{-2}$ . Hence

$$\begin{aligned}
& |\mathbf{E}[(\sigma\Psi')(X_t)J]| \\
& \leq \|\Psi(X_t)\|_{4, -k} \left\{ \|P_0(t)\sigma(X_t)J\|_{4/3, k} + \|\langle P_1(t), D(\sigma(X_t)J) \rangle_H\|_{4/3, k} \right\}.
\end{aligned}$$

Noting that  $\frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1$ , we can make estimates

$$\begin{aligned}
& \|P_0(t)\sigma(X_t)J\|_{4/3, k} \leq c't^{-\nu} \|J\|_{2, k}, \\
& \|\langle P_1(t), D(\sigma(X_t)J) \rangle_H\|_{4/3, k} \leq c't^{-\nu} \|J\|_{2, k+1}
\end{aligned}$$

for each  $t > 0$ , and for some constants  $c', \nu > 0$  (where,  $c'$  may depend on the derivatives of  $\sigma$  up to the  $(k+1)$ -th order, which are assumed to be bounded by (H1)). Now (4.6) follows.

(c) is proved similarly. The statement for  $\Lambda \in \mathcal{E}'(\mathbb{R})$  is also proved similarly.  $\square$

**Theorem 4.8.** *Let  $x \in \mathbb{R}^d$ . Suppose (H1) and (H2). Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a measurable function such that*

- (i)  *$f$  is continuous at  $x$ ,*
- (ii)  *$f \in \mathcal{E}'(\mathbb{R}^d)$ ,*
- (iii)  *$\int_0^T \|(A_i f)(X(t, x, w))\|_{2, -k}^2 dt < +\infty$  for  $i = 1, \dots, d$ ,*
- (iv)  *$\int_0^T \|(L f)(X(t, x, w))\|_{2, -k} dt < +\infty$*

for some  $k \in \mathbb{N}$ . Then we have

$$(4.7) \quad \begin{aligned} & f(X(T, x, w)) - f(x) \\ &= \sum_{i=1}^d \int_0^T (A_i f)(X(t, x, w)) dw(t) + \int_0^T (L f)(X(t, x, w)) dt \quad \text{in } \mathbb{D}^{-\infty}. \end{aligned}$$

*Proof.* By the conditions (ii), (iii), (iv) and Theorem 4.7, we have for each  $t_0 > 0$  that

$$f(X_T) - f(X_{t_0}) = \sum_{i=1}^d \int_{t_0}^T (A_i f)(X_t) dw(t) + \int_{t_0}^T (L f)(X_t) dt$$

in  $\mathbb{D}^{-\infty}$ . Now, letting  $t_0 \downarrow 0$  with using the condition (i), we obtain the result.  $\square$

By Proposition 3.10, Lemma 3.9 and Theorem 4.7, we obtain

**Corollary 4.9.** *Let  $x \in \mathbb{R}^d$  and  $\Lambda \in \mathcal{E}'(\mathbb{R}^d)$ . Assume (H1), (H2) and  $\text{supp} \Lambda \not\ni x$ . Then we have in  $\mathbb{D}^{-\infty}$ ,*

$$\Lambda(X(T, x, w)) = \sum_{i=1}^d \int_0^T (A_i \Lambda)(X(t, x, w)) dw(t) + \int_0^T (L \Lambda)(X(t, x, w)) dt.$$

**4.3. An application and examples.** In the sequel, we denote by  $X = (X_t)_{t \geq 0}$  a unique strong solution to the  $d$ -dimensional stochastic differential equation

$$(4.8) \quad dX_t = \sigma(X_t) dw(t) + b(X_t) dt, \quad X_0 = x \in \mathbb{R}^d,$$

where  $w = (w(t))_{t \geq 0}$  is a  $d$ -dimensional Wiener process. We denote by  $L$  the associated generator, i.e.,  $L = \frac{1}{2} \sum_{i,j=1}^d (\sigma^* \sigma)_j^i \partial_i \partial_j + \sum_{i=1}^d b^i \partial_i$ .

Let  $\{F_\varepsilon\}_{\varepsilon \in I} \subset \mathbb{D}^\infty(\mathbb{R}^d)$  be a bounded and uniformly non-degenerate family (see Definition 2.1). Here, the boundedness is used in the sense of  $\{F_\varepsilon\}_{\varepsilon \in I}$  is bounded in  $\mathbb{D}_p^k(\mathbb{R}^d)$  for each  $p \in (1, \infty)$  and  $k \in \mathbb{Z}_{\geq 0}$ .

**Lemma 4.10.** *For any  $s \in \mathbb{R}$ ,  $p \in (1, \infty)$  and  $p' > p$ , there exists  $c = c(s, p, p') > 0$  such that*

$$\|\Lambda(F_\varepsilon)\|_{p,s} \leq c \|(1 - \Delta)^{s/2} \Lambda\|_{L_{p'}(\mathbb{R}^d, dx)}$$

for every  $\varepsilon \in I$  and  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$  with  $(1 - \Delta)^{s/2} \Lambda \in L_{p'}(\mathbb{R}^d, dx)$ .

We note that one can take  $p' = p$  when  $F_\varepsilon = \varepsilon^{-1} w(\varepsilon^2 T)$  as mentioned in Remark A.1. A similar inequality will be proved in Subsection A.3. The same technique in the proof of Lemma A.4 is available.

Let  $H_p^s(\mathbb{R}^d) := (1 - \Delta)^{-s/2} L_p(\mathbb{R}^d, dz)$ ,  $p \in (1, \infty)$ ,  $s \in \mathbb{R}$  be the Bessel potential spaces (see [1] and [12] for details). By Lemma 4.10, given  $p \in (1, \infty)$ ,  $s \in \mathbb{R}$  and  $\Lambda \in H_p^s$ , we find the mapping  $(0, \infty) \ni t \mapsto \Lambda(X_t) \in \mathbb{D}_{p'}^s$  is continuous for  $p' \in (1, p)$  under assumptions (H1) and (H2).

**Corollary 4.11.** *Assume (H1), (H3) and (H4). Let  $p \in (1, \infty)$ ,  $s \in \mathbb{R}$ . Then for each  $\Lambda \in H_p^s(\mathbb{R}^d)$ , we have*

- (i)  $\Lambda(X_t) \in \mathbb{D}_{p'}^s$  for  $t > 0$  and  $p' \in (1, p)$ ;
- (ii) if  $p > 2$ , we further have  $\int_{t_0}^T \Lambda(X_t) dt \in \mathbb{D}_{p'}^{s+1}$  for  $t_0 \in (0, T]$  and  $p' \in [2, p)$ .

By the above remark, one would see that we can take  $p' = p$  when  $\sigma = (\text{identity matrix})$  and  $b = 0$ , i.e.,  $X_t = w(t)$ .

*Proof.* (i) is clear by Lemma 4.10.

(ii) The operator  $L$  is strongly uniformly elliptic operator of the second order because of (H3) and (H4). Hence the elliptic regularity theorem (see e.g., [1, Chapter III, Section 7.3, Theorem 7.13]) assures that  $f := (1-L)^{-1}\Lambda \in H_p^{s+2}$  and  $\sigma_j^k \partial_k f \in H_p^{s+1}$  for each  $j, k = 1, \dots, d$ . On the other hand, Theorem 4.7 gives

$$f(X_T) - f(X_{t_0}) = \sum_{j,k=1}^d \int_{t_0}^T (\sigma_j^k \partial_k f)(X_t) dw^j(t) + \int_{t_0}^T (Lf)(X_t) dt.$$

Let  $p' \in [2, p)$  be arbitrary. We find that  $f(X_T) \in \mathbb{D}_{p'}^{s+2}$  by (i) and  $\int_{t_0}^T (\sigma_j^k \partial_k f)(X_t) dw^j \in \mathbb{D}_{p'}^{s+1}$  for  $T > 0$  by Theorem 4.2, so that we have  $\int_{t_0}^T (Lf)(X_t) dt \in \mathbb{D}_{p'}^{s+1}$ . Furthermore, since  $[t_0, T] \ni t \mapsto f(X_t) \in \mathbb{D}_{p'}^{s+2}$  is continuous, this mapping is Bochner integrable and  $\int_{t_0}^T f(X_t) dt \in \mathbb{D}_{p'}^{s+2}$ . Therefore

$$\int_{t_0}^T \Lambda(X_t) dt = \int_{t_0}^T f(X_t) dt - \int_{t_0}^T (Lf)(X_t) dt \in \mathbb{D}_{p'}^{s+1}.$$

□

In the next, we investigate the class  $\mathbb{D}_p^s$  to which  $\int_{t_0}^T \Lambda(X_t) dt$  belongs when  $t_0 = 0$  for several cases of  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ .

**Example 4.1.** Assume  $d = 1$ , (H1) and (H3). Let

$$\begin{aligned} s(x) &:= \int_0^x \exp \left\{ - \int_0^z \frac{2b(\eta)}{\sigma(\eta)^2} d\eta \right\} dz, \\ m(x) &:= 2 \int_0^x \exp \left\{ \int_0^z \frac{2b(\eta)}{\sigma(\eta)^2} d\eta \right\} \frac{dz}{\sigma(z)^2}, \end{aligned} \quad \text{for } x \in \mathbb{R}.$$

The function  $s(x)$  is called the *scale function* of  $L$  and the measure  $m(dx) = m'(x)dx$  is called the *speed measure* of  $L$ . We fix  $y \in \mathbb{R}$  and define  $u : \mathbb{R} \rightarrow \mathbb{R}$  by

$$(4.9) \quad u(x) :=: u(x, y) := \frac{m'(y)}{2} |s(x) - s(y)|, \quad x \in \mathbb{R}.$$

Then it is easily checked that  $Lu = \delta_y$  and

$$(Au)(x) = \text{sgn}(x - y) \left[ \exp \left\{ - \int_y^x \frac{2b(\eta)}{\sigma(\eta)^2} d\eta \right\} \frac{\sigma(x)}{\sigma(y)^2} \right], \quad x \in \mathbb{R}.$$

in the distributional sense. Now we shall prove

$$(4.10) \quad \begin{cases} s < \frac{1}{2}; \\ p \in (1, \frac{1}{s}) \end{cases} \Rightarrow \begin{cases} 1_{\{X(t,x,w) < y\}} \in \mathbb{D}_p^s; \\ \limsup_{t \downarrow 0} \|1_{\{X(t,x,w) < y\}}\|_{p,s} < \infty. \end{cases}$$

In fact, let  $s \in (0, 1/2)$  and  $p \in (1, \frac{1}{s})$ . Take  $p' > p$  so that  $sp' < 1$ . Then by Proposition 3.5 and Lemma A.4, there exists  $c > 0$  such that

$$\begin{aligned} \|1_{\{X(t,x,w) < y\}}\|_{p,s} &= \|1_{(-\infty, (y-x)/\sqrt{t})}(\tilde{F}_t)\|_{p,s} \\ &\leq c\|(z^2 - \Delta)^{s/2}1_{(-\infty, (y-x)/\sqrt{t})}\|_{L_{p'}(\mathbb{R}, dz)} \quad \text{for all } t \in (0, T]. \end{aligned}$$

where  $\tilde{F}_t = (X^{\sqrt{t}}(1, x, w) - x)/\sqrt{t}$ . Hence we can conclude (4.10) by Proposition A.3.

By the condition (H1), we see that  $\sigma$  and  $b$  have at most linear growth, and hence  $u \in \mathcal{E}'(\mathbb{R})$ . Hence by Theorem 3.3 and Theorem 4.8, we have

$$\int_0^T \delta_y(X_t) dt = u(X_T) - u(x) - \int_0^T (Au)(X_t) dw(t).$$

It is easy to see that  $u(X_T) \in \cap_{p>1} \mathbb{D}_p^1$  and by (H3) that

$$\exp\left\{-\int_y^{X_t} \frac{2b(\eta)}{\sigma(\eta)^2} d\eta\right\} \frac{\sigma(X_t)}{\sigma(y)^2} \in \mathbb{D}_p^\infty \quad \text{for all } p \in (1, \infty).$$

From this and (4.10), we find  $\limsup_{t \downarrow 0} \|(Au)(X_t)\|_{p,s} < \infty$  for  $s < 1/2$  and  $p \in (1, \frac{1}{s})$ . Then by Theorem 4.2-(ii),  $\int_0^T (Au)(X_t) dw(t) \in \mathbb{D}_p^s$  for  $s < \frac{1}{2}$  and  $p \in [2, \frac{1}{s})$ , and thus we reached: under (H1) and (H3),

$$\int_0^T \delta_y(X_t) dt \in \mathbb{D}_p^s \quad \text{for } s < \frac{1}{2} \text{ and } p \in [2, \frac{1}{s}).$$

See Airault-Ren-Zhang [2] for a more general and stronger result.

**Remark 4.3.** In particular, we see from Example 4.1 that  $\int_0^T \delta_y(X_t) dt$  is a classical Wiener functional when  $d = 1$ , which is related to the local time at  $y$ . This can be seen as follows: Classically, the *symmetric local time*  $\{\tilde{l}(y, t) : y \in \mathbb{R}, t \geq 0\}$  is defined as a unique increasing process such that

$$|X_t - y| = |x - y| + \int_0^t \text{sgn}(X_s - y) dX_s + \tilde{l}(y, t)$$

and equivalently given by

$$\tilde{l}(y, t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(y-\varepsilon, y+\varepsilon)}(X_s) d\langle X \rangle_s$$

(see [17]). Since it holds  $\lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \sigma^2 1_{(y-\varepsilon, y+\varepsilon)} = \sigma \delta_y = \sigma(y) \delta_y$  in  $\mathcal{S}'(\mathbb{R})$ , we have

$$\tilde{l}(y, t) = \sigma(y)^2 \int_0^t \delta_y(X_u) du.$$

In the sequel, we denote  $\mathbb{D}_2^{s-} := \cap_{\varepsilon>0} \mathbb{D}_2^{s-\varepsilon}$ .

**Example 4.2.** Assume  $d = 1$ , (H1) and (H3). Let  $x, y \in \mathbb{R}$  be such that  $x \neq y$ . By Proposition 3.10, the mapping  $(0, T] \ni t \mapsto \delta'_y(X(t, x, w)) \in \mathbb{D}_2^{-k}$  is Bochner integrable for some  $k \in \mathbb{Z}_{\geq 0}$ .

Define  $u(z_1, z_2)$  by (4.9), and then

$$\begin{aligned} v(z) &:= -(\partial_{z_2} u)(z, y) \\ &= \frac{1}{2} \left\{ m'(y) s'(y) \operatorname{sgn}(z - y) - m''(y) |s(z) - s(y)| \right\} \in \mathcal{E}'(\mathbb{R}) \end{aligned}$$

satisfies  $Lv = \delta'_y$  and

$$(Av)(z) = \frac{\sigma(z)}{2} \left( 2m'(y) s'(y) \delta_y(z) - m''(y) s'(z) \operatorname{sgn}(z - y) \right).$$

Since  $X_t = X(t, x, w)$  is non-degenerate for each  $t > 0$ , it is known by Watanabe [27] that  $\delta_y(X_t) \in \mathbb{D}_p^s$  if  $s \in (-1, -\frac{1}{2})$  and  $p \in (1, \frac{1}{1+s})$ . Hence by virtue of conditions (H1) and (H3), for each  $t_0 \in (0, T)$  and  $p \geq 2$ , we find  $\int_{t_0}^T (Au)(X_t) dw(t) \in \mathbb{D}_p^{(\frac{1}{p}-1)-}$ , and thus

$$\int_{t_0}^T \delta'_y(X_t) dt \in \mathbb{D}_p^{(\frac{1}{p}-1)-} \quad \text{for any } t_0 > 0 \text{ and } p \geq 2.$$

**Example 4.3.** In the above Example 4.2, we shall try to investigate the class to which  $\int_0^T \delta'_y(X_t) dt$  belongs. Let  $p \geq 2$  be arbitrary. We notice that

$$\sigma(X_t)(m's')(y) \delta_y(X_t) = (\sigma m's')(y) \delta_y(X_t) = (\sigma m's')(y) \varepsilon^{-1} \delta_{(y-x)/\varepsilon}(\tilde{F}_\varepsilon),$$

where  $\varepsilon := \sqrt{t}$ ,  $X_t = X(t, x, w)$  and  $\tilde{F}_\varepsilon = (X_t - x)/\varepsilon$ . We shall prove

$$(4.11) \quad \varepsilon^{-1} \|\delta_{(y-x)/\varepsilon}(\tilde{F}_\varepsilon)\|_{p,s} \rightarrow 0 \quad \text{as } \varepsilon = \sqrt{t} \downarrow 0 \text{ for every } s < -1.$$

Note that the probability density function  $p_{\tilde{F}_\varepsilon}$  of  $\tilde{F}_\varepsilon$  has an estimate  $\sup_{0 < \varepsilon \leq 1} p_{\tilde{F}_\varepsilon}(z) \leq Cp(z)$  for some  $C > 0$ , where  $p(z) := (2\pi)^{-1/2} e^{-z^2/2}$ . By refining the argument in the proof of Lemma A.4, one finds that for every  $p' > p$ ,

$$\|\delta_{(y-x)/\varepsilon}(\tilde{F}_\varepsilon)\|_{p,s} \leq \text{const.} \|(1 - \Delta)^{s/2} \delta_{(y-x)/\varepsilon}\|_{L_{p'}(\mathbb{R}, \mu)}$$

for all  $\varepsilon \in (0, 1]$ , where  $\mu(dz) = p(z)dz$ . To estimate the last quantity, we apply the Fourier transformation and integration by parts which gives

$$\begin{aligned} \varepsilon^{-2} (1 - \Delta)^{s/2} \delta_{(y-x)/\varepsilon}(z) &= \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi z}}{(1 + \xi^2)^{-s/2}} \varepsilon^{-2} e^{i\xi \frac{y-x}{\varepsilon}} d\xi \\ &= \frac{1}{2\pi(y-x)^2} \int_{-\infty}^{\infty} \left\{ \frac{d^2}{d\xi^2} \frac{e^{i\xi z}}{(1 + \xi^2)^{-s/2}} \right\} e^{i\xi \frac{y-x}{\varepsilon}} d\xi, \end{aligned}$$

in which, one finds that there exists  $c = c(s) > 0$  such that

$$\left| \frac{d^2}{d\xi^2} \frac{e^{i\xi z}}{(1 + \xi^2)^{-s/2}} \right| \leq c(1 + z^2)(1 + \xi^2)^{s/2}.$$

Hence we obtain

$$|\varepsilon^{-2}(1 - \Delta)^{s/2} \delta_{(y-x)/\varepsilon}(z)| \leq \frac{c}{2\pi(y-x)^2} (1 + z^2) \int_{-\infty}^{\infty} (1 + \xi^2)^{s/2} d\xi,$$

Note that  $\int_{-\infty}^{\infty} (1 + \xi^2)^{s/2} d\xi < +\infty$  because of  $s < -1$ . We thus have

$$\sup_{\varepsilon > 0} \varepsilon^{-2p'} \|(1 - \Delta)^{s/2} \delta_{(y-x)/\varepsilon}\|_{L_{p'}(\mathbb{R}, \mu)}^{p'} \leq \text{const.} \int_{\mathbb{R}} (1 + z^2)^{p'} p(z) dz < +\infty,$$

which gives (4.11).

Therefore by Theorem 4.2-(i), we find that  $\int_0^T (Au)(X_t) dw(t) \in \mathbb{D}_p^{(-1)-}$ , and hence by Theorem 4.8,

$$\int_0^T \delta'_y(X_t) dt \in \mathbb{D}_p^{(-1)-} \quad \text{for any } p \geq 2.$$

**Example 4.4** (Cauchy's principal value of  $1/x$ ). Let  $d = 1$ . The tempered distribution  $\text{p.v.} \frac{1}{x}$  is defined by

$$\langle \text{p.v.} \frac{1}{x}, f \rangle := \lim_{\varepsilon \downarrow 0} \int_{|x| > \varepsilon} \frac{f(x)}{x} dx, \quad f \in \mathcal{S}(\mathbb{R}).$$

Let  $\Lambda \in \mathcal{S}'(\mathbb{R})$  be the regular distribution given by  $\Lambda = x \log |x| - x$ . Then the distributional derivatives are:  $\Lambda' = \log |x|$  and  $\Lambda'' = \text{p.v.} \frac{1}{x}$ .

We shall first show the Bochner integrability of the pull-back of  $(\text{p.v.} \frac{1}{x})$  by a Brownian motion  $w(t)$ :  $(0, T] \ni t \mapsto (\text{p.v.} \frac{1}{x})(w(t))$ . For each  $J \in \mathbb{D}^\infty$  and  $p, q > 1$  such that  $1/p + 1/q = 1$ , we obtain

$$\begin{aligned} \mathbf{E}[(\text{p.v.} \frac{1}{x})(w(t))J] &= \mathbf{E}[(x \log |x| - x)''(w(t))J] \\ &= \mathbf{E}[(w(t) \log |w(t)| - w(t))l_t(J)] \\ &\leq \| (w(t) \log |w(t)| - w(t)) \|_p \| l_t(J) \|_q. \end{aligned}$$

Here  $l_t(J) = \sum_{i=0}^2 \langle P_i(t), D^i J \rangle_{H^{\otimes i}} \in \mathbb{D}^\infty$  for some  $P_i(t) \in \mathbb{D}^\infty(H^{\otimes i})$ ,  $i = 0, 1, 2$  which are polynomials in  $w(t)$ , its derivatives and  $\|Dw(t)\|_H^{-2} = t^{-1}$ . Since we have

$$\begin{aligned} \|w(t) \log |w(t)|\|_p^p &= \int_{\mathbb{R}} |(\sqrt{t}y) \log(\sqrt{t}|y|)|^p \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\ &\leq 2^p |\sqrt{t} \log \sqrt{t}|^p \int_{\mathbb{R}} |y|^p \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx + 2^p t^{p/2} \int_{\mathbb{R}} |y \log |y||^p \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx, \end{aligned}$$

which tends to zero as  $t \downarrow 0$ , and hence  $\limsup_{t \downarrow 0} \|(\text{p.v.} \frac{1}{x})(w(t))\|_{p,-2} < +\infty$  for  $p \geq 2$ . This proves that  $(0, T] \ni t \mapsto (\text{p.v.} \frac{1}{x})(w(t)) \in \mathbb{D}_2^{-2}$  is Bochner integrable. On the other hand, it is clear that  $\int_0^T \|\log |w(t)|\|_2^2 dt < +\infty$ .



Now, by Theorem 4.8, we have

$$w(T) \log |w(T)| = \int_0^T \log |w(t)| dw(t) + \frac{1}{2} \int_0^T \left( \text{p.v.} \frac{1}{x} \right) (w(t)) dt.$$

By using Lemma A.1–(ii) and that  $\int_{\mathbb{R}} H_n(x) (2\pi)^{-1/2} e^{-x^2/2} dx = 0$  for  $n \geq 1$ , the chaos expansion of  $\log |w(1)|$  is computed easily and given by

$$\begin{aligned} \log |w(1)| &= \mathbf{E}[\log |w(1)|] + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \frac{1}{x} H_{n-1}(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx H_n(w(1)) \\ &= \mathbf{E}[\log |w(1)|] + \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!!}{(2n+2)!} H_{2n+2}(w(1)), \end{aligned}$$

from which, we find that

- $\|\log |w(t)|/\sqrt{t}\|_{2,s} = \|\log |w(1)|\|_{2,s} < +\infty$  iff  $s < \frac{1}{2}$ .
- A similar computation shows that

$$\|(\text{p.v.} \frac{1}{x})(w(1))\|_{2,s} < +\infty \quad \text{iff } s < -\frac{1}{2}.$$

Now, for  $s < 1/2$ , we have

$$\int_0^T \|\log |w(t)|\|_{2,s}^2 dt \leq 4 \left( \int_0^T (\log \sqrt{t})^2 dt + T \|\log |w(1)|\|_{2,s}^2 \right) < +\infty,$$

so that  $\int_0^T \log |w(t)| dw(t) \in \mathbb{D}_2^{(1/2)-}$  by Theorem 4.2–(ii). Hence

$$\int_0^T \left( \text{p.v.} \frac{1}{x} \right) (w(t)) dt \in \mathbb{D}_2^{(1/2)-}.$$

**Example 4.5.** Let  $d \geq 2$  and  $x, y \in \mathbb{R}^d$ . Suppose (H1), (H3) and (H4). Since  $\delta_y \in H_p^s$  if  $s < -(p-1)d/p$  (Lemma A.6), we find from Corollary 4.11,

$$\int_{t_0}^T \delta_y(X_t) dt \in \mathbb{D}_p^{(1-\frac{(p-1)d}{p})-} \quad \text{for } t_0 > 0 \text{ and } p \geq 2,$$

where  $X_t = X(t, x, w)$ .

Assume that  $x \neq y$ , and then we shall further investigate the class to which  $\int_0^T \delta_y(X_t) dt$  belongs. Let  $f := (1-L)^{-1} \delta_y$  and let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^\infty$ -function such that (1)  $y \notin \text{supp} \phi$ , (2)  $\phi \equiv 1$  on a neighbourhood of  $x$ , and (3)  $\text{supp} \phi$  is compact. By Theorem 4.8, we have

$$(\phi f)(X_t) - (\phi f)(x) = \sum_{i,j=1}^{\infty} \int_0^T (\sigma_j^i \partial_i (\phi f))(X_t) dw^j(t) + \int_0^T (L(\phi f))(X_t) dt,$$

in which we note that

$$L(\phi f) = (L\phi)f + \langle \sigma \nabla \phi, \sigma \nabla f \rangle_{\mathbb{R}^d} + \phi f - \delta_y$$

and  $(L\phi)f, \langle \sigma \nabla \phi, \sigma \nabla f \rangle_{\mathbb{R}^d} \in \mathcal{S}(\mathbb{R}^d)$ . By Lemma A.7–(ii),

$$\begin{cases} p \in (1, \infty); \\ -\frac{d}{2} < s < \min\{\frac{p}{p-1} - d, 0\} \end{cases} \Rightarrow \begin{cases} \lim_{t \downarrow 0} \|(\phi f)(X_t)\|_{p,s} = 0; \\ \lim_{t \downarrow 0} \|\partial_i(\phi f)(X_t)\|_{p,s} = 0, \end{cases}$$

and then we have  $\lim_{t \downarrow 0} \|(\sigma_j^i \partial_i(\phi f))(X_t)\|_{p,s} = 0$ . Now by Theorem 4.2, we obtain  $\int_0^T (\sigma_j^i \partial_i(\phi f))(X_t) dw^j(t) \in \mathbb{D}_p^s$  for  $p \geq 2$ , so that

$$\int_0^T \delta_y(X_t) dt \in \mathbb{D}_p^{(\frac{p}{p-1}-d)-} \quad \text{for any } d \geq 2 \text{ and } p \geq 2.$$

**Remark 4.4.** In Examples 4.2 and 4.5, the conclusions are seemingly not the best possible given  $p \geq 2$ , because it is natural to ask that  $\int_0^T \delta'_y(X_t) dt$  and  $\int_0^T \delta_y(X_t) dt$  belong to  $\mathbb{D}_p^{(\frac{1}{p}-1)-}$  and  $\mathbb{D}_p^{(1-\frac{(p-1)d}{p})-}$ , respectively in each example. To get further results for  $p \in (1, 2)$ , one would have to investigate Theorem 4.2 for  $p \in (1, 2)$ , which we could not.

## APPENDIX A. AUXILIARY LEMMAS

**A.1. Some knowledge of Hermite polynomials.** The *Hermite polynomials*  $H_n$ ,  $n \in \mathbb{Z}_{\geq 0}$  is defined by  $H_0(x) = 1$  and  $H_n(x) := \partial^{*n} 1(x)$  for  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ , where

$$\partial^* f(x) := -f'(x) + xf(x), \quad x \in \mathbb{R}$$

for any differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . The *Hermite functions* are now defined by

$$\phi_n(x) := H_n(\sqrt{2}x)e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R} \text{ and } n \in \mathbb{Z}_{\geq 0}.$$

Some facts about  $\{H_n\}_{n=0}^\infty$  and  $\{\phi_n\}_{n=0}^\infty$  are summarized as follows.

**Lemma A.1.** (i) For each  $n \in \mathbb{Z}_{\geq 0}$ ,

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \frac{(-1)^n e^{\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (i\xi)^n e^{-\frac{\xi^2}{2}} e^{i\xi x} d\xi.$$

(ii)  $H'_n = nH_{n-1}$  and  $H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x)$ .

(iii)  $\{\frac{1}{\sqrt{n!}} H_n\}_{n \geq 0}$  is a complete orthonormal basis of  $L_2(\mathbb{R}, (2\pi)^{-1/2} e^{-x^2/2} dx)$ .

(iv)  $\{(\sqrt{\pi}n!)^{-1/2} \phi_n\}_{n \geq 0}$  is a complete orthonormal basis of  $L_2(\mathbb{R}, dx)$ .

(v)  $(1 + x^2 - \Delta)\phi_n = 2(n+1)\phi_n$  for  $n \in \mathbb{Z}_{\geq 0}$ , where  $\Delta = \frac{d^2}{dx^2}$ .

**A.2. Some knowledge of Heaviside function.** We begin with introducing some results by Bongioanni-Torrea [5]. Let  $H := x^2 - \Delta$ ,  $A := \frac{d}{dx} + x$ ,  $B := -\frac{d}{dx} + x$  be differential operators on  $\mathcal{S}(\mathbb{R})$ . From the point of view of Lemma A.1–(iii), (iv), A family of linear operators  $(\lambda + H)^s$ ,  $s \in \mathbb{R}$  and  $\lambda \geq 0$  can be defined by the relation  $(\lambda + H)^s \phi_n = (2(n + \frac{\lambda+1}{2}))^s \phi_n$  for  $n \in \mathbb{Z}_{\geq 0}$ .

Then for  $s < 0$ , the operator  $H^{s/2}$  has an integral representation (see [5, Proposition 2])

$$(A.1) \quad (H^{s/2}f)(x) = \int_{\mathbb{R}} K_{s/2}(x, y)f(y)dy \quad \text{for } f \in \mathcal{S}(\mathbb{R}),$$

where the kernel  $K_{s/2}(x, z)$  has an estimate  $K_{s/2}(x, y) \leq c\Phi_{s/2}(|x - y|)$  for some constant  $c > 0$  and

$$\Phi_{s/2}(x) = \begin{cases} |x|^{-(1+s)}1_{\{|x|<1\}} + e^{-x^2/4}1_{\{|x|\geq 1\}} & \text{if } s > -1, \\ (1 - \log|x|)1_{\{|x|<1\}} + e^{-x^2/4}1_{\{|x|\geq 1\}} & \text{if } s = 1, \\ 1_{\{|x|<1\}} + e^{-x^2/4}1_{\{|x|\geq 1\}} & \text{if } s < -1. \end{cases}$$

**Lemma A.2.** *If  $s \in (-1, -\frac{1}{2})$  and  $p \in (1, \frac{1}{1+s})$ ,  $H^{s/2}\delta_y(x) \in L_p(\mathbb{R}, dx)$ .*

*Proof.* Since  $s > -1$ , we have

$$\begin{aligned} (H^{s/2}\delta_y)(x) &= K_{s/2}(x, y) \\ &\leq c\left(|x - y|^{-(1+s)}1_{\{|x-y|<1\}} + e^{-\frac{|x-y|^2}{4}}1_{\{|x-y|\geq 1\}}\right). \end{aligned}$$

Hence we can easily conclude the result.  $\square$

By using [5, Theorem 4, Lemma 4, Theorem 7] and  $L_p$ -multiplier theorem (see Thangavelu [21, Chapter 4, Section 4.2, Theorem 4.2.1]) for operators  $[H^{-1}(H + 2)]^{s/2}$ ,  $s \in \mathbb{R}$ , we can deduce that for each  $s \in \mathbb{R}$ , there exists a constant  $c_1 = c_1(s) > 0$  such that

$$\begin{aligned} &\|H^{(s+1)/2}f\|_{L_p(\mathbb{R}, dx)} \\ &\leq c_1\left(\|H^{s/2}Af\|_{L_p(\mathbb{R}, dx)} + \|xH^{s/2}f\|_{L_p(\mathbb{R}, dx)} + \|H^{s/2}f\|_{L_p(\mathbb{R}, dx)}\right) \end{aligned}$$

for all  $f = f(x) \in \mathcal{S}(\mathbb{R})$ . Hence, with writing  $Af =: \phi$ , we obtain

$$(A.2) \quad \begin{aligned} &\|H^{(s+1)/2}f\|_{L_p(\mathbb{R}, dx)} \\ &\leq c_1\left(\|H^{s/2}\phi\|_{L_p(\mathbb{R}, dx)} + \|xH^{s/2}f\|_{L_p(\mathbb{R}, dx)} + \|H^{s/2}f\|_{L_p(\mathbb{R}, dx)}\right). \end{aligned}$$

By a standard argument, this inequality extends and is still valid for  $f = f(x) \in \mathcal{S}'(\mathbb{R})$  such that  $(H^{s/2}Af), (H^{s/2}f), (xH^{s/2}f) \in L_p(\mathbb{R}, dx)$ .

**Proposition A.3.** *Let  $y \in \mathbb{R}$ . For each  $s < 1/2$  and  $p \in (1, 1/s)$ , we have*

- (i)  $H^{s/2}1_{(-\infty, y)} \in L_p(\mathbb{R}, dx)$ ,
- (ii)  $\lim_{y \rightarrow -\infty} \|H^{s/2}1_{(-\infty, y)}\|_{L_p(\mathbb{R}, dx)} = 0$  and
- (iii)  $\sup_{y \in \mathbb{R}} \|H^{s/2}1_{(-\infty, y)}\|_{L_p(\mathbb{R}, dx)} < \infty$ .

*Proof.* Let  $f(x) := 1_{(-\infty, y)}(x)$  and  $\phi(x) := Af(x) = \delta_y(x) + x1_{(-\infty, y)}(x)$ . Then by (A.2), we have

$$\begin{aligned} & \|H^{s/2}1_{(-\infty, y)}\|_{L_p(\mathbb{R}, dx)} \\ & \leq 2c_1 \left( \|H^{(s-1)/2}\delta_y\|_{L_p(\mathbb{R}, dx)} + \|H^{-(s-1)/2}(x1_{(-\infty, y)})\|_{L_p(\mathbb{R}, dx)} \right. \\ & \quad \left. + \|xH^{-(s-1)/2}1_{(-\infty, y)}\|_{L_p(\mathbb{R}, dx)} + \|H^{-(s-1)/2}1_{(-\infty, y)}\|_{L_p(\mathbb{R}, dx)} \right), \end{aligned}$$

which is finite by virtue of Lemma A.2 and the integral expression (A.1). Furthermore again by (A.1), we find also that the last four terms converge to zero as  $y \rightarrow -\infty$ , and uniformly bounded in  $y \in \mathbb{R}$ .  $\square$

**A.3. Fractional inequalities.** Let  $\{F_\varepsilon\}_{\varepsilon \in I} \subset \mathbb{D}^\infty$  be a bounded and uniformly non-degenerate family (see Definition 2.1). Here, the boundedness is used in the sense of  $\{F_\varepsilon\}_{\varepsilon \in I}$  is bounded in  $\mathbb{D}_p^k$  for each  $p \in (1, \infty)$  and  $k \in \mathbb{Z}_{\geq 0}$ .

**Lemma A.4.** *For any  $s \in \mathbb{R}$ ,  $p \in (1, \infty)$  and  $p' > p$ , there exists  $c = c(s, p, p') > 0$  such that*

$$\|\Lambda(F_\varepsilon)\|_{p, s} \leq c \|(x^2 - \Delta)^{s/2}\Lambda\|_{L_{p'}(\mathbb{R}, dx)}$$

for every  $\varepsilon \in I$  and  $\Lambda \in \mathcal{S}'(\mathbb{R})$  with  $(x^2 - \Delta)^{s/2}\Lambda \in L_{p'}(\mathbb{R}, dx)$ .

See also Remark A.1 just after Lemma A.5. The following proof is based on the technique in [27].

*Proof.* We show in the case of  $-2 \leq s \leq 1$ . Other cases are similarly proved. Define a linear operator  $T_\alpha(\varepsilon)$  for  $-2 \leq \alpha \leq 1$  and  $\varepsilon \in I$  by

$$T_\alpha(\varepsilon)\phi := (I - \mathcal{L})^{\alpha/2}[(x^2 - \Delta)^{-\alpha/2}\phi](F_\varepsilon)$$

for  $\phi = \phi(x) \in \mathcal{S}(\mathbb{R})$ . Since  $\{F_\varepsilon\}_{\varepsilon \in I}$  is uniformly non-degenerate, the density function  $p_{F_\varepsilon}(x)$  is uniformly bounded in  $(\varepsilon, x) \in I \times \mathbb{R}$ :

$$c_0 := \sup_{\varepsilon \in I} \sup_{x \in \mathbb{R}} p_{F_\varepsilon}(x) < \infty.$$

Take  $p' > p > 1$  arbitrary. We divide the proof into four steps.

(a) Firstly, when  $\alpha = -2$ , we shall show that  $T_{-2}(\varepsilon) : L_{p'}(\mathbb{R}, dx) \rightarrow \mathbb{D}_p^0 = L_p$  and is a continuous linear operator with an estimate

$$\|T_{-2}(\varepsilon)\|_{L_{p'}(\mathbb{R}, dx) \rightarrow L_p} \leq c_1 \quad \text{for any } \varepsilon \in I$$

for some constant  $c_1 > 0$ . In fact, let  $\phi \in \mathcal{S}(\mathbb{R})$  be arbitrary. Then

$$\|T_{-2}(\varepsilon)\phi\|_p = \|(I - \mathcal{L})^{-1}[(x^2 - \Delta)\phi](F_\varepsilon)\|_p = \|(H\phi)(F_\varepsilon)\|_{p, -2},$$

where  $H := x^2 - \Delta$ , and for each  $J \in \mathbb{D}^\infty$  and  $p'' \in (p, p')$ , we have

$$\mathbf{E}[(H\phi)(F_\varepsilon)J] = \mathbf{E}[(x^2 - \Delta)\phi](F_\varepsilon)J \leq \|\phi(F_\varepsilon)\|_{p''} \|l_\varepsilon(J)\|_{q''},$$

where  $1/p'' + 1/q'' = 1$  and  $l_\varepsilon(J)$  is of the form  $l_\varepsilon(J) = \sum_{i=0}^2 \langle P_i(\varepsilon, w), D^i J \rangle_{H^{\otimes i}}$  for some  $P_i(\varepsilon, w) \in \mathbb{D}^\infty(H^{\otimes i})$ ,  $i = 0, 1, 2$ , all of which are polynomials in  $F_\varepsilon$ , its derivatives up to the second order and  $\|DF_\varepsilon\|_H^{-2}$ . Since  $\{F_\varepsilon\}_{\varepsilon \in I}$

is non-degenerate, there exists  $c'_0 > 0$  such that  $\|l_\varepsilon(J)\|_{q''} \leq c'_0 \|J\|_{q,2}$ . Hence we have  $\mathbf{E}[(H\phi)(F_\varepsilon)J] \leq c'_0 \|\phi(F_\varepsilon)\|_{p''} \|J\|_{q,2} \leq c_0 c'_0 \|\phi\|_{L_{p'}(\mathbb{R}, dx)} \|J\|_{q,2}$  for each  $J \in \mathbb{D}^\infty$ , which implies  $\|T_{-2}(\varepsilon)\phi\|_p \leq c_1 \|\phi\|_{L_{p'}(\mathbb{R}, dx)}$ , where  $c_1 := c_0 c'_0$ . Since  $\mathcal{S}(\mathbb{R})$  is dense in  $L_{p'}(\mathbb{R}, dx)$ , we obtain the desired estimate.

(b) Next, we focus on the case of  $\alpha = 1$ . We shall show that this operator actually defines a continuous linear mapping  $T_1(\varepsilon) : L_{p'}(\mathbb{R}, dx) \rightarrow \mathbb{D}_p^0 = L_p$  with an estimate

$$\|T_1(\varepsilon)\|_{L_{p'}(\mathbb{R}, dx) \rightarrow L_p} \leq c_2 \quad \text{for any } \varepsilon \in I,$$

for some constant  $c_2 > 0$ . Let  $\phi = \phi(x) \in \mathcal{S}(\mathbb{R})$  be arbitrary. Then

$$\|T_1(\varepsilon)\phi\|_{L_p} = \|(I - \mathcal{L})^{1/2}[(x^2 - \Delta)^{-1/2}\phi](F_\varepsilon)\|_{L_p} = \|(H^{-1/2}\phi)(F_\varepsilon)\|_{p,1}.$$

By Meyer's inequality, there exists a positive constant  $c'_2 > 0$  such that  $\|J\|_{p,1} \leq c'_2 (\|J\|_{L_p} + \|DJ\|_{L_p(H)})$  for every  $J \in \mathbb{D}_2^1$ . Hence we have

$$\|(H^{-1/2}\phi)(F_\varepsilon)\|_{p,1} \leq c'_2 \left( \|(H^{-1/2}\phi)(F_\varepsilon)\|_{L_p} + \|(H^{-1/2}\phi)'(F_\varepsilon)DF_\varepsilon\|_{L_p(H)} \right).$$

We easily have  $\|(H^{-1/2}\phi)(F_\varepsilon)\|_{L_p} \leq c_0 \|(x^2 - \Delta)^{-1/2}\phi\|_{L_{p'}(\mathbb{R}, dx)}$ . Take  $q' \in (1, \infty)$  so that  $1/p' + 1/q' = 1/p$ . Then there exists  $c''_2 = c''_2(p', q') > 0$  such that  $\|J_1 J_2\|_{L_p(H)} \leq c''_2 \|J_1\|_{L_{p'}} \|J_2\|_{L_{q'}(H)}$  for all  $(J_1, J_2) \in L_{p'} \times L_{q'}(H)$ . Hence we have

$$\begin{aligned} & \|(H^{-1/2}\phi)'(F_\varepsilon)DF_\varepsilon\|_{L_p(H)} \\ & \leq c''_2 \|[(x^2 - \Delta)^{-1/2}\phi]'(F_\varepsilon)\|_{L_{p'}} \|DF_\varepsilon\|_{L_{q'}(H)} \\ & \leq c''_2 c_0 \left( \sup_{\varepsilon \in I} \|DF_\varepsilon\|_{L_{q'}(H)} \right) \|[(x^2 - \Delta)^{-1/2}\phi]'\|_{L_{p'}(\mathbb{R}, dx)}. \end{aligned}$$

By a result by Bongioanni-Torrea [5, Lemma 3 and Theorem 4], there exists a constant  $c'''_2 = c'''_2(p') > 0$  such that

$$\begin{aligned} & \|[(x^2 - \Delta)^{-1/2}\phi]'\|_{L_{p'}(\mathbb{R}, dx)} + \|(x^2 - \Delta)^{-1/2}\phi\|_{L_{p'}(\mathbb{R}, dx)} \\ & \leq \left\| \left( \frac{d}{dx} + x \right) (x^2 - \Delta_z)^{-1/2}\phi \right\|_{L_{p'}(\mathbb{R}, dx)} + \|(x^2 - \Delta_z)^{-1/2}\phi\|_{L_{p'}(\mathbb{R}, dx)} \\ & \quad + \|x(x^2 - \Delta)^{-1/2}\phi\|_{L_{p'}(\mathbb{R}, dx)} \\ & \leq c'''_2 \|\phi\|_{L_{p'}(\mathbb{R}, dx)}, \end{aligned}$$

and hence we have obtained

$$\|T_1(\varepsilon)\phi\|_{L_p} \leq c_2 \|\phi\|_{L_{p'}(\mathbb{R}, dx)} \quad \text{for all } \varepsilon \in I$$

as desired, where  $c_2 := c_0 c'_2 (1 + c''_2 \sup_{\varepsilon \in I} \|D\tilde{F}_\varepsilon\|_{L_{q'}(H)}) c'''_2$ .

(c) For each  $\varepsilon \in I$ , the family of operators  $\{T_\alpha(\varepsilon)\}_{-2 < \alpha < 1}$  poses the analytic continuation  $T_z(\varepsilon)$  to the strip  $-2 < \operatorname{Re}(z) < 1$  in the sense that  $\Phi(z) = \mathbf{E}[(T_z(\varepsilon)\phi)\psi]$  is analytic on  $-2 < \operatorname{Re}(z) < 1$  and continuous on  $-2 \leq \operatorname{Re}(z) \leq 1$  for any  $\phi \in L_{p'}(\mathbb{R}, dx)$  and  $\psi \in L_q =$

$\mathbb{D}_2^0$ , where  $1/p + 1/q = 1$ . Moreover, the complex function  $\Phi$  has the estimate

$$\sup_{\tau \in \mathbb{R}} \sup_{-2 \leq \alpha \leq 1} |\Phi(\alpha + i\tau)| < +\infty$$

for each  $\phi$  and  $\psi$ . By the Marcinkiewicz  $L_p$ -multiplier theorem (see e.g. [21, Chapter 4, Section 4.2, Theorem 4.2.1]), one gets  $\sup_{\tau \in \mathbb{R}} \|(x^2 - \Delta)^{i\tau}\|_{L_{p'}(\mathbb{R}, dx) \rightarrow L_{p'}(\mathbb{R}, dx)} < +\infty$ . On the other hand, by Meyer's  $L_p$ -multiplier theorem (see e.g. [11, Chapter V, Section 8, Lemma 8.2]) gives  $\sup_{\tau \in \mathbb{R}} \|(I - \mathcal{L})^{i\tau}\|_{L_p \rightarrow L_p} < +\infty$ . These can be used to check that  $\sup_{\tau \in \mathbb{R}} \|T_{-2+i\tau}(\varepsilon)\|_{L_{p'}(\mathbb{R}, dx) \rightarrow L_p} < +\infty$  and  $\sup_{\tau \in \mathbb{R}} \|T_{1+i\tau}(\varepsilon)\|_{L_{p'}(\mathbb{R}, dx) \rightarrow L_p} < +\infty$ . Then by Stein's interpolation theorem (see [20, Theorem 1]), we can conclude that

$$\sup_{\varepsilon \in I} \sup_{-2 \leq \alpha \leq 1} \|T_\alpha(\varepsilon)\|_{L_{p'}(\mathbb{R}, dx) \rightarrow L_p} < +\infty.$$

(d) Now, for each  $\phi \in \mathcal{S}(\mathbb{R})$ , we have

$$\begin{aligned} \|\phi(F_\varepsilon)\|_{p,s} &= \|(I - \mathcal{L})^{s/2}[(x^2 - \Delta)^{-s/2}(x^2 - \Delta)^{s/2}\phi](F_\varepsilon)\|_{L_p} \\ &= \|T_s(\varepsilon)[(x^2 - \Delta)^{s/2}\phi]\|_{L_p} \\ &\leq \|T_s(\varepsilon)\|_{L_{p'}(\mathbb{R}, dx) \rightarrow L_p} \|(x^2 - \Delta)^{s/2}\phi\|_{L_{p'}(\mathbb{R}, dx)}. \end{aligned}$$

Again by denseness of  $\mathcal{S}(\mathbb{R})$ , this inequality extends to  $\Lambda \in \mathcal{S}'(\mathbb{R})$  such that  $(x^2 - \Delta)^{s/2}\Lambda \in L_{p'}(\mathbb{R}, dx)$ .  $\square$

In a similar vein, we can prove the following: Let  $\{F_\varepsilon\}_{\varepsilon \in I} \subset \mathbb{D}^\infty(\mathbb{R}^d)$  be a bounded and uniformly non-degenerate family.

**Lemma A.5.** *For any  $s \in \mathbb{R}$ ,  $p \in (1, \infty)$  and  $p' > p$ , there exists  $c = c(s, p, p') > 0$  such that*

$$\|\Lambda(F_\varepsilon)\|_{p,s} \leq c \|(1 - \Delta)^{s/2}\Lambda\|_{L_{p'}(\mathbb{R}^d, dx)}$$

for every  $\varepsilon \in I$  and  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$  with  $(1 - \Delta)^{s/2}\Lambda \in L_{p'}(\mathbb{R}^d, dx)$ .

**Remark A.1.** In Lemma A.4 and Lemma A.5, one can take  $p' = p$  when  $F_\varepsilon = \varepsilon^{-1}w(\varepsilon^2T)$ , where  $\varepsilon \in (0, T] =: I$ ,  $T > 0$  and  $w = (w^1, \dots, w^d)$  is the canonical process, i.e., the  $d$ -dimensional Wiener process starting at zero (assume  $d = 1$  if considering Lemma A.4), because then  $DF_\varepsilon = \varepsilon^{-1}Dw(\varepsilon^2T) = \varepsilon^{-1}(1_{[0, \varepsilon^2T]}, \dots, 1_{[0, \varepsilon^2T]})$  and  $\langle DF_\varepsilon^i, DF_\varepsilon^j \rangle_H = \delta_{ij}T$  are non-random.

**A.4. Regularity of something related to resolvent kernel associated with elliptic operator.** Suppose (H1), (H3) and (H4). We set, for  $f \in \mathcal{S}(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$ ,

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^d (\sigma^* \sigma)_j^i(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b^i(x) \frac{\partial f}{\partial x_i}(x).$$

The fractional power  $(1 - \Delta)^{s/2}$  is defined as a pseudo-differential operator:

$$(1 - \Delta)^{s/2} \phi(x) = \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s/2} \widehat{\phi}(\xi) e^{i\langle \xi, x \rangle} d\xi, \quad \phi \in \mathcal{S}(\mathbb{R}^d),$$

where  $\widehat{\phi}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \phi(y) e^{-i\langle \xi, y \rangle} dy$  is the Fourier transform of  $\phi$ , and is also given by

$$(1 - \Delta)^{s/2} \phi(x) = \frac{1}{\Gamma(-s/2)} \int_0^\infty t^{-\frac{s}{2}-1} e^{-t} (e^{t\Delta} \phi)(x) dt, \quad \phi \in \mathcal{S}(\mathbb{R}^d)$$

for  $-d < s < 0$ , where  $e^{t\Delta}$  is the heat semigroup associated to  $\Delta$ . Actually, formulae  $\phi(y) = \int_{\mathbb{R}^d} \widehat{\phi}(\xi) e^{i\langle \xi, y \rangle} d\xi$  and  $\int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4t}} e^{i\langle \xi, y \rangle} dy = (4\pi t)^{d/2} e^{-|\xi|^2 t}$  give the above equivalence.

Before entering the following series of estimates, we recall that

$$\int_{|x|<1} |x|^{-s} dx < +\infty \quad \text{iff} \quad s < d,$$

where the integral is over  $\mathbb{R}^d$  and  $|x| = |x|_{\mathbb{R}^d}$ .

**Lemma A.6.** *For every  $p \in (1, \infty)$  and  $s < -(p-1)d/p$ , we have  $(1 - \Delta)^{s/2} \delta_y \in L_p(\mathbb{R}^d, dx)$  for each  $y \in \mathbb{R}^d$ .*

*Proof.* The transition density associated to  $\Delta$  is given by  $p_t(x, y) = (4\pi t)^{-d/2} \exp(-|y-x|^2/(4t))$ . Hence we have

$$\begin{aligned} (1 - \Delta)^{s/2} \delta_y(x) &= \frac{1}{\Gamma(-s/2)} \int_0^\infty t^{-\frac{s}{2}-1} e^{-t} p_t(x, y) dt \\ &= \frac{(4\pi)^{-d/2}}{\Gamma(-s/2)} \int_0^\infty t^{-\frac{s+d}{2}-1} e^{-t} \exp\left\{-\frac{|x-y|^2}{4t}\right\} dt \\ &= \frac{(4\pi)^{-d/2}}{4^{-\frac{s+d}{2}} \Gamma(-s/2)} |x-y|^{-(d+s)} \int_0^\infty u^{\frac{d+s}{2}-1} e^{-u} \exp\left\{-\frac{|x-y|^2}{4u}\right\} du, \end{aligned}$$

which behaves, up to a multiplicative constant, as  $|x-y|^{-(d+s)}$  when  $|x-y| \rightarrow 0$  and rapidly decreasing as  $|x-y| \rightarrow \infty$ . Therefore,  $(1 - \Delta)^{s/2} \delta_y \in L_p(\mathbb{R}^d, dx)$  if  $(d+s)p < d$ , i.e.,  $s < -(p-1)d/p$ .  $\square$

Let  $x \in \mathbb{R}^d$ . Under the conditions (H1), (H3) and (H4), we denote by  $X_t = X(t, x, w)$  a unique strong solution to the stochastic differential equation (4.8). In the sequel, we fix  $y \in \mathbb{R}^d$  such that  $y \neq x$  and define

$$f_y(z) := (1 - L)^{-1} \delta_y(z), \quad z \in \mathbb{R}^d.$$

Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^\infty$ -function such that (1)  $y \notin \text{supp} \phi$ , (2)  $\phi \equiv 1$  on a neighbourhood of  $x$ , and (3)  $\text{supp} \phi$  is compact.

**Lemma A.7.** *Let  $p \in (1, \infty)$  be arbitrary.*

- (i) *For each  $s < \min\{1 - \frac{(p-1)d}{p}, 0\}$  and  $t > 0$ , we have  $(\nabla f_y)(X_t) \in \mathbb{D}_p^s(\mathbb{R}^d)$  and  $(0, T] \ni t \mapsto (\nabla f_y)(X_t) \in \mathbb{D}_p^s(\mathbb{R}^d)$  is continuous.*

- (ii) Assume  $d \geq 2$ . Then for each  $-\frac{d}{2} < s < \min\{\frac{p}{p-1} - d, 0\}$ , we have  $\lim_{t \rightarrow 0} \|(\nabla(\phi f_y))(X_t)\|_{p,s} = 0$  and  $\lim_{t \rightarrow 0} \|(\phi f_y)(X_t)\|_{p,s} = 0$ .

*Proof.* Let  $p \in (1, \infty)$  and  $t_0 > 0$  be arbitrary. In the following, we write  $f := f_y$ .

(i) Take  $p' > p$  such that  $s < \min\{1 - \frac{(p'-1)d}{p'}, 0\}$ . Since  $\{X_t\}_{t_0 \leq t \leq T}$  is uniformly non-degenerate, the assertion follows by Lemma A.5 once we show  $(1 - \Delta)^{s/2}(\partial f / \partial z_k) \in L_{p'}(\mathbb{R}^d, dz)$ .

Denote by  $p_t(z, z')$  the transition density associated to  $L$ . By a standard estimate (see e.g., [9, Chapter 9, Section 6, Theorem 7]), there exist  $c, C > 0$  such that

$$\begin{aligned} p_t(z, z') &\leq C(2\pi ct)^{-d/2} \exp\left\{-\frac{|z - z'|^2}{2ct}\right\}, \\ \left|\frac{\partial p_t}{\partial z_k}(z, z')\right| &\leq Ct^{-1/2}(2\pi ct)^{-d/2} \exp\left\{-\frac{|z - z'|^2}{2ct}\right\} \end{aligned}$$

for every  $k = 1, \dots, d$  and  $z, z' \in \mathbb{R}^d$ . We may assume  $c \geq 2$  by rearranging  $C > 0$ . Then we have

$$\begin{aligned} \left|\frac{\partial f}{\partial z_k}(z)\right| &\leq \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-t} \left|\frac{\partial p_t}{\partial z_k}(z, y)\right| dt \\ &\leq \frac{C}{\Gamma(1/2)} \int_0^\infty t^{-1/2} e^{-t} (2\pi ct)^{-d/2} \exp\left\{-\frac{|z - y|^2}{2ct}\right\} dt, \end{aligned}$$

so that

$$\begin{aligned} &\left|[(1 - \Delta)^{s/2} \frac{\partial f}{\partial z_k}](z)\right| \\ &= \frac{1}{\Gamma(-s/2)\Gamma(1/2)} \int_0^\infty t^{-\frac{s}{2}-1} e^{-t} \int_{\mathbb{R}^d} \frac{e^{-\frac{|z-z'|^2}{4t}}}{\sqrt{4\pi t}^d} \frac{\partial f}{\partial z_k}(z') dz' dt \\ &\leq \frac{C'}{\Gamma(-s/2)\Gamma(1/2)} \int_0^\infty \int_0^\infty t^{-\frac{s}{2}-1} u^{-1/2} e^{-(t+u)} \frac{e^{-\frac{|z-y|^2}{2c(t+u)}}}{\sqrt{2\pi c(t+u)}^d} du dt \\ &= \frac{C'}{\Gamma(\frac{1-s}{2})} \int_0^\infty v^{\frac{1-s}{2}-1} e^{-v} \frac{e^{-\frac{|z-y|^2}{2cv}}}{\sqrt{2\pi cv}^d} dv, \end{aligned}$$

for some constant  $C' > 0$ . Hence  $(1 - \Delta)^{s/2}(\partial_k f)(z) \in L_{p'}(\mathbb{R}^d, dz)$ .

(ii) Suppose that  $s < \min\{\frac{p}{p-1} - d, 0\}$ . Take  $\delta > 0$  so that  $\{z \in \mathbb{R}^d : |z - y| < \delta\} \subset (\text{supp } \phi)^c$ . Note that  $\partial_k(\phi f_y) = (\partial_k \phi) f_y + \phi(\partial_k f_y)$  and  $(\partial_k \phi) f_y \in \mathcal{S}(\mathbb{R}^d)$ . Therefore  $\lim_{t \downarrow 0} \|[(\partial_k \phi) f_y](X_t)\|_{p,s} = 0$  is clear. So in the following, we investigate the behaviour of  $(\phi(\partial_k f_y))(X_t)$  and  $(\phi f_y)(X_t)$ . For this, we divide the proof into four steps.



(a) Since  $s \leq 0$ , we notice that

$$\begin{aligned} |(I - \mathcal{L})^{s/2} F| &= \left| \frac{1}{\Gamma(-\frac{s}{2})} \int_0^\infty u^{-\frac{s}{2}-1} e^{-u} T_u F du \right| \\ &\leq \frac{1}{\Gamma(-\frac{s}{2})} \int_0^\infty u^{-\frac{s}{2}-1} e^{-u} T_u |F| du = (I - \mathcal{L})^{s/2} |F| \end{aligned}$$

for any  $F \in L_2$ , where  $T_u = \exp(u\mathcal{L})$ ,  $u \geq 0$  is the Ornstein-Uhlenbeck semigroup on the Wiener space. Then, taking  $p', q, r > 1$  so that  $\frac{1}{p'} + \frac{1}{q} + \frac{1}{r} < 1$  and with putting  $F := (\phi \partial_k f)(X_t) = (\phi \partial_k f)(X_t) 1_{\{|X_t - x| > \delta\}}$ , we have

$$\begin{aligned} \|(\phi \partial_k f)(X_t)\|_{p,s}^p &= \|(I - \mathcal{L})^{s/2} F\|_p^p \\ &\leq \mathbf{E} \left[ |(I - \mathcal{L})^{s/2} F|^{p-1} (I - \mathcal{L})^{s/2} |F| \right] \\ &= \mathbf{E} \left[ |(\phi \partial_k f)(X_t)| \left\{ (I - \mathcal{L})^{s/2} |(I - \mathcal{L})^{s/2} F|^{p-1} \right\} 1_{\{|X_t - x| > \delta\}} \right] \\ &\leq c_0 \|(\phi \partial_k f)(X_t)\|_{p'} \| (I - \mathcal{L})^{s/2} F \|_{q,s}^{p-1} \mathbf{P}(|X_t - x| > \delta)^{1/r}, \end{aligned}$$

where  $c_0 = c_0(p', q, r) > 0$  is a constant independent of  $t$ . We easily have  $\|(\phi \partial_k f)(X_t)\|_{p'} \leq |\phi|_\infty \|(\partial_k f)(X_t)\|_{p'}$  and

$$\begin{aligned} &\| |(I - \mathcal{L})^{s/2} F|^{p-1} \|_{q,s} \\ &\leq \| \{ (I - \mathcal{L})^{s/2} F \}^{p-1} \|_q \\ &= \| (I - \mathcal{L})^{s/2} F \|_{q(p-1)}^{p-1} = \| (I - \mathcal{L})^{s/2} (\phi \partial_k f)(X_t) \|_{p''}^{p-1}, \end{aligned}$$

where  $p'' := q(p-1)$ . Thus we have obtained

$$\begin{aligned} &\|(\phi \partial_k f)(X_t)\|_{p,s}^p \\ \text{(A.3)} \quad &\leq c_0 |\phi|_\infty \|(\partial_k f)(X_t)\|_{p'} \times \| (I - \mathcal{L})^{s/2} (\phi \partial_k f)(X_t) \|_{p''}^{p-1} \\ &\quad \times \mathbf{P}(|X_t - x| > \delta)^{1/r}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\|(\phi f)(X_t)\|_{p,s}^p \\ \text{(A.4)} \quad &\leq c'_0 \|(\phi f)(X_t)\|_{p'} \times \| (I - \mathcal{L})^{s/2} (\phi f)(X_t) \|_{p''}^{p-1} \\ &\quad \times \mathbf{P}(|X_t - x| > \delta)^{1/r} \end{aligned}$$

for some constant  $c_0 > 0$ , independent of  $t$ .

(b) We will write  $\varepsilon := \sqrt{t}$  in the sequel. We shall give estimates for each factors in (A.3) and (A.4), though the proof is presented in the next step. Note that for the last factor in both, there exists  $c_3, c'_3, K > 0$  such that

$$\text{(A.5)} \quad \mathbf{P}(|X_t - x| > \delta) \leq c_3 \exp \left( - \frac{\delta^2}{c'_3 \varepsilon^2} \right) \quad \text{for } t = \varepsilon^2 \in (0, K].$$

Let  $p \in (1, \infty)$  anew be arbitrary. Then for (A.3), we shall prove

$$(A.6) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon^d \|(\partial_k f)(X_t)\|_p^p < +\infty \quad \text{if } p < \frac{d}{d-1},$$

$$(A.7) \quad \limsup_{t \downarrow 0} \varepsilon^{d-sp} \|(I - \mathcal{L})^{s/2}(\phi \partial_k f)(X_t)\|_p^p < +\infty$$

$$\text{if } p < \frac{d}{d+s-1}.$$

On the other hand, for (A.4), we shall prove

$$(A.8) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon^d \|f(X_t)\|_{L_p(\mathbb{R}^d, dz)}^p < +\infty \quad \text{if } p < \frac{d}{d-1},$$

$$(A.9) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon^{d-s} \|(I - \mathcal{L})^{s/2}(\phi f)(X_t)\|_p^p < +\infty$$

$$\text{if } p < \min \left\{ \frac{d}{d+s-1}, \frac{2d}{d-2} \right\} = \frac{d}{d+s-1},$$

where  $\frac{2d}{d-2}$  is understood as  $+\infty$  if  $d = 2$  and the last equality is because of  $s > -\frac{d}{2}$ .

(c) Take  $p' > p$  arbitrary. First we shall prove (A.6). By using Lemma A.5, we have

$$\begin{aligned} \|(\partial_k f)(X_t)\|_p^{p'} &\leq c_1 \|(\partial_k f)(x + \varepsilon z)\|_{L_{p'}(\mathbb{R}^d, dz)}^{p'} \\ &\leq c'_1 \int_{\mathbb{R}^d} \left\{ \int_0^\infty u^{-1/2} e^{-u} (2\pi c u)^{-d/2} \exp \left\{ -\frac{|\varepsilon z - (y-x)|^2}{2cu} \right\} du \right\}^{p'} dz \\ &\leq c'_1 \varepsilon^{-d} \int_{\mathbb{R}^d} \left\{ |z - (y-x)|^{(1-d)} \int_0^\infty u^{\frac{d-1}{2}-1} e^{-u} e^{-\frac{|z-(y-x)|^2}{2cu}} du \right\}^{p'} dz. \end{aligned}$$

for some constants  $c_1, c'_1 > 0$  independent of  $\varepsilon$ . Here, the assumption  $d \geq 2$  assures  $\int_0^\infty u^{\frac{d-1}{2}-1} e^{-u} du < +\infty$  and hence (A.6) follows.

For (A.7), with assuming  $\varepsilon \in (0, 1]$ , we have

$$\begin{aligned}
& (1 - \Delta)^{s/2}[(\phi \partial_k f)(x + \varepsilon \bullet)](z) \\
&= \frac{1}{\Gamma(-\frac{s}{2})} \int_0^\infty u^{-\frac{s}{2}-1} e^{-u} \int_{\mathbb{R}^d} \frac{e^{-\frac{|z-z'|^2}{4u}}}{\sqrt{4\pi u}^d} (\phi \partial_k f)(x + \varepsilon z') dz' du \\
&\leq c_2 \int_0^\infty \int_0^\infty u^{-\frac{s}{2}-1} e^{-u} v^{-1/2} e^{-v} \frac{e^{-\frac{|\varepsilon z - (y-x)|^2}{2c(\varepsilon^2 u + v)}}}{\sqrt{2\pi c(\varepsilon^2 u + v)}^d} dudv \\
&= c_2 \varepsilon^s \int_0^\infty \int_0^\infty u^{-\frac{s}{2}-1} v^{-1/2} e^{-(\varepsilon^2 u + v)} \frac{e^{-\frac{|\varepsilon z - (y-x)|^2}{2c(u+v)}}}{\sqrt{2\pi c(u+v)}^d} dudv \\
&\leq c'_2 \varepsilon^s \int_0^\infty \int_0^\infty u^{-\frac{s}{2}-1} v^{-1/2} e^{-(u+v)} \frac{e^{-\frac{|\varepsilon z - (y-x)|^2}{2c(u+v)}}}{\sqrt{2\pi c(u+v)}^d} dudv \\
&= c'_2 \varepsilon^s \int_0^\infty u^{\frac{1-s}{2}-1} e^{-u} \frac{e^{-\frac{|\varepsilon z - (y-x)|^2}{2cu}}}{\sqrt{2\pi cu}^d} du,
\end{aligned}$$

for some constants  $c_2, c'_2 > 0$  independent of  $\varepsilon$ , and so that, by using Lemma A.5,

$$\begin{aligned}
& \|(I - \mathcal{L})^{s/2}(\phi \partial_k f)(X_t)\|_p^{p'} \\
&\leq c_2'' \|(1 - \Delta)^{s/2}[(\partial_k f)(x + \varepsilon \bullet)](z)\|_{L_{p'}(\mathbb{R}^d, dz)}^{p'} \\
&\leq c_2''' \varepsilon^{sp'} \int_{\mathbb{R}^d} \left\{ \int_0^\infty u^{\frac{1-s}{2}-1} e^{-u} \frac{e^{-\frac{|\varepsilon z - (y-x)|^2}{2cu}}}{\sqrt{2\pi cu}^d} du \right\}^{p'} dz \\
&= c_2''' \varepsilon^{sp'-d} \int_{\mathbb{R}^d} \left\{ |z - (y-x)|^{-(d-(1-s))} \right. \\
&\quad \left. \times \int_0^\infty u^{\frac{d-(1-s)}{2}-1} e^{-u} e^{-\frac{|z-(y-x)|^2}{2cu}} du \right\}^{p'} dz.
\end{aligned}$$

for some constants  $c_2'', c_2''' > 0$ . Here, note that  $d - (1 - s) > 0$  because of  $d \geq 2$  and  $s > -\frac{d}{2}$ . Hence  $\int_0^\infty u^{\frac{d-(1-s)}{2}-1} e^{-u} du < +\infty$  and (A.7) follows.

Next we prove (A.8). By virtue of Lemma A.5, it suffices to show

$$\begin{aligned}
& (A.10) \\
& \|f(x + \varepsilon z)\|_{L_{p'}(\mathbb{R}^d, dz)}^{p'} \\
&\leq c_1 \varepsilon^{-d} \left[ \int_{\mathbb{R}^d} \left\{ |z - (x-y)|^{1-d} \int_0^\infty u^{\frac{d}{2}-1} e^{-u} e^{-\frac{|z-(x-y)|^2}{2cu}} du \right\}^{p'} dz \right. \\
&\quad \left. + \int_{\mathbb{R}^d} \left\{ |z - (x-y)|^{-(d-1)/2} \int_0^\infty u^{-1/2} e^{-u} e^{-\frac{|z-(x-y)|^2}{2cu}} du \right\}^{p'} dz \right]
\end{aligned}$$

for some constant  $c_1 > 0$ , independent of  $\varepsilon$ . To prove this, we begin with the inequality

$$f(x + \varepsilon z) \leq \frac{C(2\pi c)^{-d/2}}{\Gamma(1)} \int_0^\infty u^{-d/2} e^{-u} e^{-\frac{|y-(x+\varepsilon z)|^2}{2cu}} du.$$

We divide the integral as

$$\begin{aligned} & \int_0^\infty u^{-d/2} e^{-u} e^{-\frac{|y-(x+\varepsilon z)|^2}{2cu}} du \\ &= \int_0^{|y-(x+\varepsilon z)|} u^{-d/2} e^{-u} e^{-\frac{|y-(x+\varepsilon z)|^2}{2cu}} du + \int_{|y-(x+\varepsilon z)|}^\infty u^{-d/2} e^{-u} e^{-\frac{|y-(x+\varepsilon z)|^2}{2cu}} du. \end{aligned}$$

The first term in the last equation is estimated as

$$\begin{aligned} & \int_0^{|y-(x+\varepsilon z)|} u^{-d/2} e^{-u} e^{-\frac{|y-(x+\varepsilon z)|^2}{2cu}} du \\ &= (2c)^{\frac{d}{2}-1} |\varepsilon z - (x - y)|^{2-d} \int_{\frac{|y-(x+\varepsilon z)|}{2c}}^\infty u^{\frac{d}{2}-2} e^{-u} e^{-\frac{|\varepsilon z - (x - y)|^2}{2cu}} du \\ &\leq (2c)^{\frac{d}{2}} |\varepsilon z - (x - y)|^{1-d} \int_{\frac{|y-(x+\varepsilon z)|}{2c}}^\infty u^{\frac{d}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x - y)|^2}{2cu}} du \\ &\leq (2c)^{\frac{d}{2}} |\varepsilon z - (x - y)|^{1-d} \int_0^\infty u^{\frac{d}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x - y)|^2}{2cu}} du. \end{aligned}$$

On the other hand, the second term is

$$\begin{aligned} & \int_{|y-(x+\varepsilon z)|}^\infty u^{-d/2} e^{-u} e^{-\frac{|y-(x+\varepsilon z)|^2}{2cu}} du \\ &\leq |\varepsilon z - (x - y)|^{-(d-1)/2} \int_0^\infty u^{-1/2} e^{-u} e^{-\frac{|y-(x+\varepsilon z)|^2}{2cu}} du. \end{aligned}$$

Thus a change of variable leads us to (A.10) and (A.8) is proved.

For (A.9), it is sufficient to prove

$$\begin{aligned} & (A.11) \\ & \| (1 - \Delta)^{s/2} [(\phi f)(x + \varepsilon \bullet)](z) \|_{L_{p'}(\mathbb{R}^d, dz)}^{p'} \\ & \leq c_2 \varepsilon^{-s-d} \left[ \int_{\mathbb{R}^d} \left\{ |z - (x - y)|^{1-(s+d)} \int_0^\infty u^{\frac{d+s}{2}-1} e^{-u} e^{-\frac{|z-(x-y)|^2}{2cu}} du \right\}^{p'} dz \right. \\ & \quad \left. + \int_{\mathbb{R}^d} \left\{ |z - (x - y)|^{\frac{2-d}{2}} \int_0^\infty u^{-\frac{s}{2}} e^{-u} e^{-\frac{|z-(x-y)|^2}{2cu}} du \right\}^{p'} dz \right] \end{aligned}$$

where  $c_2 > 0$  is a constant independent of  $\varepsilon$ . Note that  $d + s > 0$  and  $-s > 0$ , so that it is assured that  $\int_0^\infty u^{\frac{d+s}{2}-1} e^{-u} du < +\infty$  and  $\int_0^\infty u^{-\frac{s}{2}} e^{-u} du < +\infty$ , respectively. Note also that  $1 - (s + d) < 0$ , and hence (A.11) enables us to conclude (A.9) by a use of Lemma A.5.

To prove (A.11), we apply a similar argument, with assuming  $\varepsilon \in (0, 1]$ , which leads to

$$\begin{aligned} & |(1 - \Delta)^{s/2}[(\phi f)(x + \varepsilon \bullet)](z)| \\ & \leq \text{const.} \varepsilon^s \int_0^\infty u^{\frac{2-(s+d)}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du. \end{aligned}$$

We divide the integral as

$$\begin{aligned} & \int_0^\infty u^{\frac{2-(s+d)}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du \\ & = \int_0^{|\varepsilon z - (x-y)|} u^{\frac{2-(s+d)}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du \\ & \quad + \int_{|\varepsilon z - (x-y)|}^\infty u^{\frac{2-(s+d)}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du. \end{aligned}$$

We estimate the first term as follows.

$$\begin{aligned} & \int_0^{|\varepsilon z - (x-y)|} u^{\frac{2-(s+d)}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du \\ & = (2c)^{1-\frac{d+s-2}{2}} |\varepsilon z - (x-y)|^{2-(s+d)} \int_{\frac{|\varepsilon z - (x-y)|}{2c}}^\infty u^{\frac{d+s-2}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du \\ & \leq (2c)^{1-\frac{d+s-3}{2}} |\varepsilon z - (x-y)|^{1-(s+d)} \int_{\frac{|\varepsilon z - (x-y)|}{2c}}^\infty u^{\frac{d+s}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du \\ & \leq (2c)^{1-\frac{d+s-3}{2}} |\varepsilon z - (x-y)|^{1-(s+d)} \int_0^\infty u^{\frac{d+s}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du. \end{aligned}$$

The second term is estimated as

$$\begin{aligned} & \int_{|\varepsilon z - (x-y)|}^\infty u^{\frac{2-(s+d)}{2}-1} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du \\ & \leq |\varepsilon z - (x-y)|^{\frac{2-d}{2}} \int_0^\infty u^{-\frac{s}{2}} e^{-u} e^{-\frac{|\varepsilon z - (x-y)|^2}{2cu}} du. \end{aligned}$$

Combining these, and a change of variable, we reach the estimate (A.11), and hence (A.9) is proved.

(d) From the view of (A.6), (A.7), (A.8) and (A.9), what we have to do now is to find  $p', q, r > 1$  such that

- $\frac{1}{p'} + \frac{1}{q} + \frac{1}{r} < 1$ ;
- $p' < \frac{d}{d-1}$  and  $p'' := q(p-1) < \frac{d}{d-(1-s)}$ .

In fact, since  $s < \frac{p}{p-1} - d$ , we can take  $\varepsilon \in (0, \frac{1}{d})$  such that  $s < \frac{p}{p-1} - \frac{d}{1-\varepsilon d} < \frac{p}{p-1} - d$ . Then take  $p', q > 1$  so that

$$\frac{d-1}{d} < \frac{1}{p'} < \frac{d-1}{d} + \varepsilon \quad \text{and} \quad \frac{1}{q} = \frac{1}{d} - \varepsilon \quad \left( < \frac{1}{d} \right).$$

These conditions imply  $1/p' + 1/q < \frac{d-1}{d} + \frac{1}{d} = 1$  and hence one can take  $r > 0$  such that  $1/p' + 1/q + 1/r < 1$ . Finally, we have  $1 - \frac{(p''-1)d}{p''} = 1 - \frac{(p''-1)d}{(p-1)q} > 1 - \frac{(p''-1)d}{(p-1)d} = 1 - \frac{p''-1}{p-1} = \frac{p}{p-1} - q = \frac{p}{p-1} - \frac{d}{1-\varepsilon d} > s$ , which implies  $p'' < \frac{d}{d-(1-s)}$ .

Therefore, by taking  $p', q, r > 1$  as above, we conclude from (A.3), (A.4) and (A.5)–(A.9) that  $\|(\phi \partial_k f)(X_t)\|_{p,s}$  and  $\|(\phi f)(X_t)\|_{p,s}$  converge to zero as  $t = \varepsilon^2 \downarrow 0$ .

□

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